TOPOLOGICAL DESCRIPTION OF RIEMANNIAN FOLIATIONS WITH DENSE LEAVES

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Introduction

Riemannian foliations occupy an important place in geometry. An excellent survey is A. Haefliger's Bourbaki seminar [6], and the book of P. Molino [13] is the standard reference for riemannian foliations. In one of the appendices to this book, E. Ghys proposes the problem of developing a theory of equicontinuous foliated spaces paralleling that of riemannian foliations; he uses the suggestive term "qualitative riemannian foliations" for such foliated spaces.

In our previous paper [1], we discussed the structure of equicontinuous foliated spaces and, more generally, of equicontinuous pseudogroups of local homeomorphisms of topological spaces. This concept was difficult to develop because of the nature of pseudogroups and the failure of having an infinitesimal characterization of local isometries, as one does have in the riemannian case. These difficulties give rise to two versions of equicontinuity: a weaker version seems to be more natural, but a stronger version is more useful to generalize topological properties of riemannian foliations. Another relevant property for this purpose is quasi-effectiveness, which is a generalization to pseudogroups of effectiveness for group actions. In the case of locally connected foliated spaces, quasi-effectiveness is equivalent to the quasi-analyticity introduced by

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Haefliger [4]. For instance, the following well-known topological properties of riemannian foliations were generalized to strongly equicontinuous quasi-effective compact foliated spaces [1]; let us remark that we also assume that all foliated spaces are locally compact and polish:

- Leaves without holonomy are quasi-isometric to one another (our original motivation).
- Leaf closures define a partition of the space. So the foliated space is transitive (there is a dense leaf) if and only if it is minimal (all leaves are dense).
- The holonomy pseudogroup has a closure defined by using the compact-open topology on small enough open subsets.

In this paper, we show, in fact, that there are few ways of constructing nice equicontinuous foliated spaces beyond riemannian foliations. The definition of riemannian foliation used here is slightly more general than usual: a foliation is called riemannian when its holonomy pseudogroup is given by local isometries of some riemannian manifold (a riemannian pseudogroup); thus leafwise smoothness is not required. Our main result is the following purely topological characterization of riemannian foliations with dense leaves on compact manifolds.

Theorem. *Let $(X, \mathfrak{F})$ be a transitive compact foliated space. Then $\mathfrak{F}$ is a riemannian foliation if and only if X is locally connected and finite dimensional, $\mathfrak{F}$ is strongly equicontinuous and quasi-analytic, and the closure of its holonomy pseudogroup is quasi-analytic.*

This theorem is a direct consequence of the corresponding result for pseudogroups, whose proof uses the material developed in [1] as well as the local version of the solution of Hilbert 5th problem due to R. Jacoby [9].

An earlier result in this direction was that of M. Kellum [10, 11] who proved this property for certain pseudogroups of uniformly Lipschitz diffeomorphisms of riemannian manifolds. Also, R. Sacksteder work [17] can be interpreted as giving a characterization of riemannian pseudogroups of one-dimensional manifolds. Another similar result, proved by C. Tarquini [19], states that equicontinuous transversely conformal foliations are riemannian; note that, in the case of dense leaves, this result of Tarquini follows easily from our main theorem.

1. Local groups and local actions

The concept of local group and allied notions is developed in Jacoby [9]. Some of these notions are recalled in this section, for ease of reference.

Definition 1.1. A local group is a quintuple $(G, e, \cdot, ', \mathfrak{D})$ satisfying the following conditions:

(1) $(G, \mathfrak{D})$ is a topological space;
(2) $\cdot$ is a function from a subset of $G \times G$ to G;
(3) $'$ is a function from a subset of G to G;
(4) there is a subset O of G such that
  (a) O is an open neighborhood of e in G,
  (b) $O \times O$ is a subset of the domain of $\cdot$.
  (c) O is a subset of the domain of $'$,
  (d) for all $a, b, c \in O$, if $a \cdot b$ and $b \cdot c \in O$, then $(a \cdot b) \cdot c = (a \cdot b) \cdot c$.

(e) for all $a \in O$, $a' \in O$, $a \cdot e = e \cdot a = a$ and $a' \cdot a = a \cdot a' = e$,
(f) the map $\cdot : O \times O \to G$ is continuous,
(g) the map $' : O \to G$ is continuous;

(5) the set $\{e\}$ is closed in G.

Jacoby employs the notation $\mathfrak{G}$ for the quintuple $(G, e, \cdot, ', \mathfrak{D})$, but here it will be simply denoted by G.

The collection of all sets O satisfying condition (4) will be denoted by ΨG. This is a neighborhood base of $e \in G$; all of these neighborhoods are symmetric with respect to the inverse operation (3). Let $\Phi(G, n)$ denote the collection of subsets A of G such that the product of any collection of $\leq n$ elements of A is defined, and the set A^n of such products is contained in some $O \in \Psi G$.

If G is a local group, then H is a subgroup of G if $H \in \Phi(G, 2)$, $e \in H$, $H' = H$ and $H \cdot H = H$.

If G is a local group, then $H \subset G$ is a sub-local group of G in case H is itself a local group with respect to the induced operations and topology.

If G is a local group, then ΥG denotes the set of all pairs (H, U) of subsets of G so that (1) $e \in H$; (2) $U \in \Psi G$; (3) for all $a, b \in U \cap H$, $a \cdot b \in H$; (4) for all $c \in U \cap H$, $c' \in H$.

Jacoby [9, Theorem 26] proves that $H \subset G$ is a sub-local group if and only if there exists U such that $(H, U) \in \Upsilon G$.

Let G be a local group and let ΠG denote the pairs (H, U) so that (1) $e \in H$; (2) $U \in \Psi G \cap \Phi(G, 6)$; (3) for all $a, b \in U^6 \cap H$, $a \cdot b \in H$; (4) for all $c \in U^6 \cap H$, $c' \in H$; (5) $U^2 \setminus H$ is open. Given such a pair $(H, U) \in \Pi G$, there is a (completely regular, hausdorff) topological space $G/(U, H)$ and a continuous open surjection

$$T : U^2 \to G/(U, H)$$

such that $T(a) = T(b)$ if and only if $a' \cdot b \in H$ (*cf.* [9, Theorem 29]).

If (H, V) is another pair in ΠG, then the spaces $G/(H, U)$ and $G/(H, V)$ are locally homeomorphic in an obvious way. Thus the concept of coset space of H is well defined in this sense, as a germ of a topological space. The notation G/H will be used in this sense; and to say that G/H has certain topological property will mean that some $G/(H, U)$ has such property.

Let ΔG be the set of pairs (H, U) such that $(H, U) \in \Pi G$ and, for all $a \in H \cap U^4$ and $b \in U^2$, $b' \cdot (a \cdot b) \in H$. A subset $H \subset G$ is called a normal sub-local group of G if there exists U such that $(H, U) \in \Delta G$. If $(H, U) \in \Delta G$ then the quotient space $G/(H, U)$ admits the structure of a local group (see [9, Theorem 35] for the pertinent details) and the natural projection $T : U^2 \to G/(H, U)$ is a local homomorphism. As before, another such pair (H, V) produces a locally isomorphic quotient local group.

Let us recall the main results of Jacoby [9] on the structure of locally compact local groups because they will be needed in the sequel.

Theorem 1.2 (Jacoby [9, Theorem 96])**.** *Any locally compact local group without small subgroups is a local Lie group.*

In the above result, a local group without small subgroups is a local group where some neighborhood of the identity element contains no nontrivial subgroup.

Theorem 1.3 (Jacoby [9, Theorems 97–103])**.** *Any locally compact second countable local group G can be approximated by local Lie groups. More precisely, given $V \in \Psi G \cap$*

$\Phi(G,2)$, there exists $U \in \Psi G$ with $U \subset V$ and there exists a sequence of compact normal subgroups $F_n \subset U$ such that (1) $F_{n+1} \subset F_n$, (2) $\bigcap_n F_n = \{e\}$, (3) $(F_n, U) \in \Delta G$, and (4) $G/(F_n, U)$ is a local lie group.

Theorem 1.4 (Jacoby [9, Theorem 107])**.** *Any finite dimensional metrizable locally compact local group is locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group.*

An immediate consequence of Theorem 1.4 is that any locally euclidean local group is a local Lie group, which is the local version of Hilbert 5th problem obtained by Jacoby.

All local groups appearing in this paper will be assumed, or proved, to be locally compact and second countable.

Definition 1.5. A local group G is a *local transformation group* on a subspace $X \subset Y$ if there is given a continuous map $G \times X \to Y$, written $(g,x) \mapsto gx$, such that

- $ex = x$ for all $x \in X$; and
- $g_1(g_2x) = (g_1g_2)x$, provided both sides are defined.

This map $G \times X \to Y$ is called a *local action* of G on $X \subset Y$.

The typical example of local action is the following. Let H be a sub-local group of G. If $(H,U) \in \Pi G$ and $T : U^2 \to G/(H,U)$ is the natural projection, then U is a sub-local group of G and the map $(u, T(g)) \mapsto T(u \cdot g)$ defines a local action of U on the open subspace $T(U)$ of $G/(H,U)$.

If G is a local group acting on $X \subset Y$ and the action is locally transitive at $x \in X$ in the sense that there is a neighborhood $V \in \Psi G$ such that Vx includes a neighborhood of x in X, then there is a sub-local group H of G and an open subset $U \subset G$ such that $(H,U) \in \Pi G$ and the orbit map $g \in G \mapsto gx \in X$ induces a local homeomorphism $G/(H,U) \to X$ at x, which is equivariant with respect to the action of U.

Theorem 1.6. *Let G be a locally compact, separable and metrizable local group. Suppose that there is a local action of G on a finite dimensional subspace $X \subset Y$ and that the action is locally transitive at some $x \in X$. Fix some $(H,U) \in \Pi G$ so that the orbit map $g \mapsto gx$ induces a local homeomorphism $G/(H,U) \to X$ at x. Then there exists a connected normal subgroup K of G such that $K \subset H$, $(K,U) \in \Pi G$ and $G/(K,U)$ is finite dimensional.*

Proof. This is a local version of [15, Theorem 6.2.2], whose proof shows the following assertion that will be used now.

Claim 1. *Let A be a locally compact, separable and metrizable topological group, and let B be a closed subgroup of A such that A/B is of finite dimension and connected. Let N_n be a sequence of compact normal subgroups so that $\bigcap_n N_n = \{e\}$ and every A/N_n is a Lie group. Then there is some index n_0 such that the connected component of the identity of N_{n_0} is contained in B.*

The following observation is also needed.

Claim 2. *Let A be a local group, let $(B,V) \in \Pi A$, let $T : A \to A/(B,V)$ denote the natural projection, and let C be a compact subgroup of A contained in $V^2 \cap V^6$. Then $B \cap C$ is a compact subgroup of C, a map $C/(B \cap C) \to A/(B,V)$ is well defined by the assignment $a(B \cap C) \mapsto T(a)$, and this map is an embedding.*

This assertion can be proved as follows. On the one hand, $B \cap C$ is compact because B is closed and C compact. On the other hand, $B \cap C$ is a subgroup of C because C is a subgroup, $C \subset V^6$, and $a \cdot b \in B$ and $a' \in B$ for all $a, b \in V^6$ since $(B, V) \in \Pi A$. The map $C/(B \cap C) \to A/(B, V)$ is well defined and injective because $C \subset V^2$ and $T(a) = T(b)$ if and only if $a \cdot b' \in B$ for $a, b \in V^2$. This injection is continuous because it is induced by the inclusion $C \hookrightarrow V^2$. Thus this map is an embedding since $C/(B \cap C)$ is compact and $A/(B, V)$ is Hausdorff.

Now, with the notation of the statement of this theorem, let F_n be a sequence of compact normal subgroups of G as provided by Jacoby's theorem [9] (quoted as Theorem 1.3). It may be assumed that $(F_n, U) \in \Delta G$ and $F_n \subset U^2 \cap U^6$ for all n. If K_n is the identity component of each F_n, then the natural quotient map $G/(K_n, U) \to G/(F_n, U)$ has zero dimensional fibers, because they are locally homeomorphic to the zero-dimensional group F_n/K_n. Because each $G/(F_n, U)$ is a local Lie group, it is finite dimensional, and thus $G/(K_n, U)$ is also finite dimensional (see [8, Ch. VII, §4]).

By Claim 2, $K_1 \cap H$ is a compact subgroup of K_1, and there is a canonical embedding $K_1/(K_1 \cap H) \to G/(H, U)$. Moreover $K_1/(K_1 \cap H)$ is connected since so is K_1. Then the dimension of $K_1/(K_1 \cap H)$ is less or equal than the dimension of $G/(H, U)$ by [8, Theorem III 1], and thus $K_1/(K_1 \cap H)$ is of finite dimension. On the other hand, each canonical embedding $K_1/(K_1 \cap F_n) \to G/(F_n, U)$, given by Claim 2, realizes $K_1/(K_1 \cap F_n)$ as a compact subgroup of the local Lie group $G/(F_n, U)$ because $K_1 \cap F_n$ is a normal subgroup of K_1. So every $K_1/(K_1 \cap F_n)$ is a Lie group. Then, by Claim 1 with $A = K_1$, $B = K_1 \cap H$ and $N_n = K_1 \cap F_n$, there is some index n_0 such that the identity component K of $F = K_1 \cap F_{n_0}$ is contained in $K_1 \cap H$. This F is a normal subgroup of G, and thus K is a connected normal subgroup of G. Furthermore $(K, U), (F, U) \in \Delta G$, and

$$\dim G/(K, U) = \dim G/(F, U) \leq \dim G/(K_1, U) + \dim K_1/(K_1 \cap F_{n_0})$$

by [8, Theorem III 4]. So $G/(K, U)$ is of finite dimension as desired. □

2. Equicontinuous pseudogroups

A *pseudogroup of local transformations* of a topological space Z is a collection $\mathcal{H}$ of homeomorphisms between open subsets of Z that contains the identity on Z and is closed under composition (wherever defined), inversion, restriction and combination of maps. Such a pseudogroup $\mathcal{H}$ is *generated* by a set $E \subset \mathcal{H}$ if every element of $\mathcal{H}$ can be obtained from E by using the above pseudogroup operations; the sets of generators will be assumed to be symmetric for simplicity ($h^{-1} \in E$ if $h \in E$). The *orbit* of an element $x \in Z$ is the set $\mathcal{H}(x)$ of elements $h(x)$, for all $h \in \mathcal{H}$ whose domain contains x. These orbits are the equivalence classes of an equivalence relation on Z.

Pseudogroups of local transformations are natural generalizations of group actions on topological spaces (each group action generates a pseudogroup). Another important example of a different nature is the holonomy pseudogroup of a foliated space defined by a regular covering by flow boxes [2, 4, 5, 7].

The study of pseudogroups can be simplified by using certain equivalence relation introduced by Haefliger [4, 5]. For instance, any pseudogroup of local transformations is equivalent to its restriction to any open subset that meets all orbits; indeed, the whole of this equivalence relation is generated by this very basic type of examples. This concept of pseudogroup equivalence is very important in the study of foliated spaces because the equivalence class of the holonomy pseudogroup depends only on each foliated space; it is independent of the choice of a regular covering by flow boxes.

For a pseudogroup $\mathcal{H}$ of local transformations of a locally compact space Z, Haefliger introduced also the concept of compact generation: $\mathcal{H}$ is *compactly generated* if there is a relatively compact open set U in Z meeting each orbit of $\mathcal{H}$, and such that the restriction $\mathcal{G}$ of $\mathcal{H}$ to U is generated by a finite symmetric collection $E \subset \mathcal{G}$ so that each $g \in E$ is the restriction of an element $\bar{g}$ of $\mathcal{H}$ defined on some neighborhood of the closure of the source of g. This notion is invariant by equivalences and the relatively compact open set U meeting each orbit can be chosen arbitrarily. If E satisfies the above conditions, it is called a *system of compact generation* of $\mathcal{H}$ on U.

The concept of strong and weak equicontinuity was introduced in [1] for pseudogroups of local transformations of spaces whose topology is induced by the following type of structure. Let $\{(Z_i, d_i)\}_{i\in I}$ be a family of metric spaces such that $\{Z_i\}_{i\in I}$ is a covering of a set Z, each intersection $Z_i \cap Z_j$ is open in (Z_i, d_i) and (Z_j, d_j), and for all $\varepsilon > 0$ there is some $\delta(\varepsilon) > 0$ so that the following property holds: for all $i, j \in I$ and $z \in Z_i \cap Z_j$, there is some open neighborhood $U_{i,j,z}$ of z in $Z_i \cap Z_j$ (with respect to the topology induced by d_i and d_j) such that

$$d_i(x, y) < \delta(\varepsilon) \Longrightarrow d_j(x, y) < \varepsilon$$

for all $\varepsilon > 0$ and all $x, y \in U_{i,j,z}$. Such a family is called a *cover of Z by quasi-locally equal metric spaces*. Two such families are called *quasi-locally equal* when their union also is a cover of Z by quasi-locally equal metric spaces. This is an equivalence relation whose equivalence classes are called *quasi-local metrics* on Z. For each quasi-local metric $\mathfrak{Q}$ on Z, the pair $(Z, \mathfrak{Q})$ is called a *quasi-local metric space*. Such a $\mathfrak{Q}$ induces a topology on Z so that, for each $\{(Z_i, d_i)\}_{i\in I} \in \mathfrak{Q}$, the family of open balls of all metric spaces (Z_i, d_i) form a base of open sets. Any topological concept or property of $(Z, \mathfrak{Q})$ refers to this underlying topology. It was also observed in [1] that $(Z, \mathfrak{Q})$ is a locally compact polish space if and only if it is hausdorff, paracompact, separable and locally compact.

The strongest version of equicontinuity was defined in [1] as follows. Let $\mathcal{H}$ be a pseudogroup of local homeomorphisms of a quasi-local metric space $(Z, \mathfrak{Q})$. Then $\mathcal{H}$ is called *strongly equicontinuous* if there exists some $\{(Z_i, d_i)\}_{i\in I} \in \mathfrak{Q}$ and some symmetric set S of generators of $\mathcal{H}$ that is closed under compositions such that, for every $\varepsilon > 0$, there is some $\delta(\varepsilon) > 0$ so that

$$d_i(x, y) < \delta(\varepsilon) \Longrightarrow d_j(h(x), h(y)) < \varepsilon$$

for all $h \in S$, $i, j \in I$ and $x, y \in Z_i \cap h^{-1}(Z_j \cap \operatorname{im} h)$.

The condition on S to be closed under compositions is precisely what distinguishes strong and weak equicontinuity [1, Lemma 8.3]. A typical choice of S is the set of all possible composites of some symmetric set of generators. In fact, given any S satisfying the condition of strong equicontinuity, it is obviously possible to find a symmetric set of generators E given by restrictions of elements of S, and then the set of all composites of elements of E also satisfies the condition of strong equicontinuity.

A pseudogroup $\mathcal{H}$ acting on a space Z will be called *strongly equicontinuous* when it is strongly equicontinuous with respect to some quasi-local metric inducing the topology of Z. This notion is invariant by equivalences of pseudogroups acting on locally compact polish spaces [1, Lemma 8.8].

A key property of strong equicontinuity is the following.

Proposition 2.1 ([1, Proposition 8.9]). *Let $\mathcal{H}$ be a compactly generated and strongly equicontinuous pseudogroup acting on a locally compact polish quasi-local metric space $(Z, \mathfrak{Q})$, and let U be any relatively compact open subset of $(Z, \mathfrak{Q})$ that meets every $\mathcal{H}$-orbit. Suppose that $\{(Z_i, d_i)\}_{i\in I} \in \mathfrak{Q}$ satisfies the condition of strong equicontinuity. Let E be any system of compact generation of $\mathcal{H}$ on U, and let $\bar{g}$ be an extension of each $g \in E$ with $\overline{\operatorname{dom} g} \subset \operatorname{dom} \bar{g}$. Also, let $\{Z'_i\}_{i\in I}$ be any shrinking of $\{Z_i\}_{i\in I}$. Then there is a finite family $\mathcal{V}$ of open subsets of $(Z, \mathfrak{Q})$ whose union contains U and such that, for any $V \in \mathcal{V}$, $x \in U \cap V$, and $h \in \mathcal{H}$ with $x \in \operatorname{dom} h$ and $h(x) \in U$, the domain of $\tilde{h} = \bar{g}_n \circ \cdots \circ \bar{g}_1$ contains V for any composite $h = g_n \circ \cdots \circ g_1$ defined around x with $g_1, \dots, g_n \in E$, and moreover $V \subset Z'_{i_0}$ and $\tilde{h}(V) \subset Z'_{i_1}$ for some $i_0, i_1 \in I$.*

The following terminology was introduced in [1] to study strongly equicontinuous pseudogroups. A pseudogroup $\mathcal{H}$ of local transformations of a space Z is said to be *quasi-effective* if it is generated by some symmetric set S that is closed under compositions, and such that any transformation in S is the identity on its domain if it is the identity on some non-empty open subset of its domain. The family S can be assumed to be also closed under restrictions to open sets, and thus every map in $\mathcal{H}$ is a combination of maps in S in this case. Moreover, if $\mathcal{H}$ is strongly equicontinuous and quasi-effective, then S can be chosen to satisfy the conditions of both strong equicontinuity and quasi-effectiveness. The notion of quasi-effectiveness is invariant by equivalences of pseudogroups acting on locally compact polish spaces [1, Lemma 9.5]. Moreover this property is equivalent to quasi-analyticity for pseudogroups acting on locally connected and locally compact polish spaces [1, Lemma 9.6]; recall that a pseudogroup $\mathcal{H}$ is called *quasi-analytic* if every $h \in \mathcal{H}$ is the identity around some $x \in \operatorname{dom} h$ whenever h is the identity on some open set whose closure contains x [4].

Proposition 2.2 ([1, Proposition 9.9]). *Let $\mathcal{H}$ be a compactly generated, strongly equicontinuous and quasi-effective pseudogroup of local homeomorphisms of a locally compact polish space Z. Suppose that the conditions of strong equicontinuity and quasi-effectiveness are satisfied with a symmetric set S of generators of $\mathcal{H}$ that is closed under compositions. Let A, B be open subsets of Z such that $\overline{A}$ is compact and contained in B. If x and y are close enough points in Z, then*

$$f(x) \in A \Longrightarrow f(y) \in B$$

for all $f \in S$ whose domain contains x and y.

Recall that a pseudogroup is called *transitive* when it has a dense orbit, and is called *minimal* when all of its orbits are dense.

Theorem 2.3 ([1, Theorem 11.1]). *Let $\mathcal{H}$ be a compactly generated and strongly equicontinuous pseudogroup of local transformations of a locally compact polish space Z. If $\mathcal{H}$ is transitive, then $\mathcal{H}$ is minimal.*

For spaces Y, Z, let $C(Y, Z)$ denote the family of continuous maps $Y \to Z$, which will be denoted by $C_{\text{c-o}}(Y, Z)$ when it is endowed with the compact-open topology. For open subspaces O, P of a space Z, the space $C_{\text{c-o}}(O, P)$ will be considered as an open subspace of $C_{\text{c-o}}(O, Z)$ in the canonical way.

Theorem 2.4 ([1, Theorem 12.1]). *Let $\mathcal{H}$ be a quasi-effective, compactly generated and strongly equicontinuous pseudogroup of local transformations of a locally compact polish space Z. Let S be a symmetric set of generators of $\mathcal{H}$ that is closed under compositions* *and restrictions to open subsets, and satisfies the conditions of strong equicontinuity and quasi-effectiveness. Let $\widetilde{\mathcal{H}}$ be the set of maps h between open subsets of Z that satisfy the following property: for every $x \in \operatorname{dom} h$, there exists a neighborhood O_x of x in $\operatorname{dom} h$ so that the restriction $h|_{O_x}$ is in the closure of $C(O_x, Z) \cap S$ in $C_{\text{c-o}}(O_x, Z)$. Then:*

- *$\widetilde{\mathcal{H}}$ is closed under composition, combination and restriction to open sets;*
- *every map in $\widetilde{\mathcal{H}}$ is a homeomorphism around every point of its domain;*
- *the maps of $\widetilde{\mathcal{H}}$ that are homeomorphisms form a pseudogroup $\overline{\mathcal{H}}$ that contains $\mathcal{H}$;*
- *$\overline{\mathcal{H}}$ is strongly equicontinuous;*
- *the orbits of $\overline{\mathcal{H}}$ are equal to the closures of the orbits of $\mathcal{H}$; and*
- *$\widetilde{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are independent of the choice of S.*

If a pseudogroup $\mathcal{H}$ satisfies the conditions of Theorem 2.4, then the pseudogroup $\overline{\mathcal{H}}$ is called the *closure* of $\mathcal{H}$.

Note that a pseudogroup $\mathcal{H}$ of local transformations of a locally compact space Z is quasi-effective just when there is a symmetric set S of generators of $\mathcal{H}$ that is closed under compositions and restrictions to open subsets, and such that the restriction map $\rho^V_W : S \cap C(V, Z) \to S \cap C(W, Z)$ is injective for all open subsets V, W of Z with $W \subset V$. If moreover Z is a locally compact polish space, and $\mathcal{H}$ is compactly generated and strongly equicontinuous, then any such ρ^V_W is bijective for V, W small enough by Proposition 2.1. Moreover ρ^V_W is continuous with respect to the compact-open topology [14, p. 289], but it may not be a homeomorphism as shown by the following example.

Example 2.5. Let Z be the union of two tangent spheres in $\mathbb{R}^3$, and let $h : Z \to Z$ be the combination of two rotations, one on each sphere, around the common axis and with rationally independent angles. Then h generates a compactly generated, strongly equicontinuous and quasi-effective pseudogroup $\mathcal{H}$ of local transformations of Z; indeed, h is an isometry for the path metric space structure on Z induced from that of $\mathbb{R}^3$. Nevertheless, it is easy to see that the closure $\overline{\mathcal{H}}$ is not quasi-effective.

Lemma 2.6. *Let $\mathcal{H}$ be a compactly generated, strongly equicontinuous and quasi-effective pseudogroup of local transformations of a locally compact polish space Z. Then $\overline{\mathcal{H}}$ is quasi-effective if and only if there is a symmetric set S of generators of $\mathcal{H}$ that is closed under compositions and restrictions to open subsets, and such that $\rho^V_W : S \cap C(V, Z) \to S \cap C(W, Z)$ is a homeomorphism with respect to the compact-open topologies for small enough open subsets V, W of Z with $W \subset V$.*

Proof. The result follows directly by observing that, according to Theorem 2.4, $\overline{\mathcal{H}}$ is quasi-effective just when there is some symmetric set S of generators of $\mathcal{H}$ that is closed under compositions and satisfies the following condition: for any sequence h_n in S and open non-empty subsets V, W of Z, with $W \subset V \subset \operatorname{dom} h_n$ for all n, if $h_n|_W \to \mathrm{id}_W$ in $C_{\text{c-o}}(W, Z)$, then $h_n|_V \to \mathrm{id}_V$ in $C_{\text{c-o}}(V, Z)$. □

Corollary 2.7. *Let $\mathcal{H}$ be a compactly generated, strongly equicontinuous and quasi-analytic pseudogroup of local transformations of a locally connected and locally compact polish space Z. Then $\overline{\mathcal{H}}$ is quasi-analytic if and only if there is a symmetric set S of generators of $\mathcal{H}$ that is closed under compositions and restrictions to open subsets, and such that $\rho^V_W : S \cap C(V, Z) \to S \cap C(W, Z)$ is a homeomorphism with respect to the compact-open topologies for small enough open subsets V, W of Z with $W \subset V$.*

Finally, let us recall from [1] certain isometrization theorem, which states that equicontinuous quasi-effective pseudogroups are indeed pseudogroups of local isometries in some sense. First, two metrics on the same set are said to be *locally equal* when they induce the same topology and each point has a neighborhood where both metrics are equal. Let $\{(Z_i,d_i)\}_{i\in I}$ be a family of metric spaces such that $\{Z_i\}_{i\in I}$ is a covering of a set Z, each intersection $Z_i\cap Z_j$ is open in (Z_i,d_i) and (Z_j,d_j), and the metrics d_i,d_j are locally equal on $Z_i\cap Z_j$ whenever this is a non-empty set. Such a family will be called a *cover of Z by locally equal metric spaces*. Two such families are called *locally equal* when their union also is a cover of Z by locally equal metric spaces. This is an equivalence relation whose equivalence classes are called *local metrics* on Z. For each local metric $\mathfrak{D}$ on Z, the pair $(Z,\mathfrak{D})$ is called a *local metric space*. Observe that every metric induces a unique local metric in a canonical way. In turn, every local metric canonically determines a unique quasi-local metric. Note also that local metrics induced by metrics can be considered as germs of metrics around the diagonal. Moreover a local or quasi-local metric is induced by some metric if and only if it is hausdorff and paracompact [1, Theorems 13.5 and 15.1].

Now, a local homeomorphism h of a local metric space $(Z,\mathfrak{D})$ is called a *local isometry* if there is some $\{(Z_i,d_i)\}_{i\in I}\in\mathfrak{D}$ such that, for $i,j\in I$ and $z\in Z_i\cap h^{-1}(Z_j\cap\operatorname{im} h)$, there is some neighborhood $U_{h,i,j,z}$ of z in $Z_i\cap h^{-1}(Z_j\cap\operatorname{im} h)$ so that $d_i(x,y)=d_j(h(x),h(y))$ for all $x,y\in U_{h,i,j,z}$. This definition is independent of the choice of the family $\{(Z_i,d_i)\}_{i\in I}\in\mathfrak{D}$. Then the isometrization theorem is the following.

Theorem 2.8 ([1, Theorem 15.1])**.** *Let $\mathcal{H}$ be a compactly generated, quasi-effective and strongly equicontinuous pseudogroup of local transformations of a locally compact polish space Z. Then $\mathcal{H}$ is a pseudogroup of local isometries with respect to some local metric inducing the topology of Z.*

3. Riemannian pseudogroups

Definition 3.1. A pseudogroup $\mathcal{H}$ of local transformations of a space Z is called a *riemannian pseudogroup* if Z is a Hausdorff paracompact C^∞-manifold and all maps in $\mathcal{H}$ are local isometries with respect to some riemannian metric on Z.

Example 3.2. Let G be a local Lie group, $G_0\subset G$ a compact subgroup. Then the canonical local action of some neighborhood of the identity in G on some neighborhood of the identity class in G/G_0 generates a transitive riemannian pseudogroup. In fact, since G_0 is compact, there is a G-left invariant and G_0-right invariant riemannian metric on some neighborhood of the identity in G, which induces a G-invariant riemannian metric on some neighborhood of the identity class in G/G_0. With more generality, if $\Gamma\subset G$ a dense sub-local group, then the canonical local action of some neighborhood of the identity in Γ on some neighborhood of the identity class in G/G_0 generates a transitive riemannian pseudogroup. Moreover this riemannian pseudogroup is complete in the sense of [4]. It is well known that any transitive complete riemannian pseudogroup is equivalent to a pseudogroup of this type, which follows from the pseudogroup version of Molino description of riemannian foliations.

The pseudogroup version of the main result of this paper is the following topological characterization of transitive compactly generated riemannian pseudogroups.

Theorem 3.3. *Let $\mathcal{H}$ be a transitive, compactly generated pseudogroup of local transformations of a locally compact polish space Z. Then $\mathcal{H}$ is a riemannian pseudogroup if* *and only if Z is locally connected and finite dimensional, $\mathcal{H}$ is strongly equicontinuous and quasi-analytic, and $\overline{\mathcal{H}}$ is quasi-analytic.*

Remark. The closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ exists by virtue of Theorem 2.4, because the space Z is locally connected, hence the pseudogroup $\mathcal{H}$ is quasi-effective because it is quasi-analytic [1, Lemma 9.6].

The following is a direct consequence of the above theorem.

Corollary 3.4. *Let $\mathcal{H}$ be a compactly generated, strongly equicontinuous and quasi-analytic pseudogroup of local transformations of a locally compact polish space Z. Then the $\mathcal{H}$-orbit closures are C^∞ manifolds if and only if they are locally connected and finite dimensional, and the induced pseudogroup $\overline{\mathcal{H}}$ is quasi-analytic on them.*

Proof. This follows from Theorem 3.3 because the closure of $\mathcal{H}$ acting on the closure of an orbit is equivalent to a pseudogroup like Example 3.2. □

The proof of Theorem 3.3 will be given in the next section; in the interim, some examples illustrating the necessity of several hypotheses are described.

Example 3.5. Let Z be the product of countably infinitely many circles. This is a compact, locally connected polish group which acts on itself by translations in an equicontinuous way. Let $\mathbb{Z} \to Z$ be an injective homomorphism with dense image. Then the action of $\mathbb{Z}$ on Z induced by this homomorphism is minimal and equicontinuous, and so it generates a minimal, quasi-analytic and equicontinuous pseudogroup, which is not riemannian because Z is of infinite dimension.

Example 3.6. Let Z be the set of p-adic numbers $x \in \mathbb{Q}_p$ with p-adic norm $|x|_p \leq 1$. Then the operation $x \mapsto x + 1$ defines an action of $\mathbb{Z}$ on Z which is minimal and equicontinuous (it preserves the p-adic metric on Z). Thus it generates a minimal, quasi-analytic and equicontinuous pseudogroup, which is not riemannian because Z is zero-dimensional.

Example 3.7. A related example is as follows. Let Z be the standard cantor set in $[0, 1] \subset \mathbb{R}$ together with all integer translates. Then there is a pseudogroup $\mathcal{H}$ acting on Z which is generated by translations of the line which locally preserve Z. In fact, $\mathcal{H}$ is a pseudogroup of local isometries for two geometrically distinct metrics, the euclidean and the dyadic.

Example 3.8. The previous example can be generalized, replacing Z by the universal Menger curve [3, Ch. 15]. This space Z (to be precise, a modification of it) is constructed as an invariant set of the pseudogroup of local homeomorphisms of $\mathbb{R}^3$ generated by the map $f(x) = 3x$ and the three unit translations parallel to the coordinate axes. There is a pseudogroup acting on Z generated by euclidean isometries which locally preserve Z. It is fairly easy to see that such a pseudogroup is minimal, quasi-analytic and equicontinuous. Moreover Z is locally connected and of dimension one. However, this pseudogroup is not compactly generated.

4. Equicontinuous pseudogroups and Hilbert's 5th problem

This section is devoted to prove Theorem 3.3. The "only if" part is obvious, so it is enough to show the "if" part, which has essentially two steps. In the first one, a local group action on Z is obtained as the closure of the set of elements of $\mathcal{H}$ which are sufficiently close to the identity map on an appropriate subset of Z. This construction follows Kellum [10]. The second step invokes the theory behind the solution to the local version of Hilbert's 5th problem in order to show that the local group is a local Lie group, and thus this local action is isometric for some riemannian metric if its isotropy subgroups are compact. So $\mathcal{H}$ is proved to be riemannian by showing that it is of the type described in Example 3.2.

By Theorem 2.8, there is a local metric structure $\mathfrak{D}$ on Z with respect to which the elements of $\mathcal{H}$ are local isometries. Take any $\{(Z_i, d_i)\}_{i\in I} \in \mathfrak{D}$ satisfying the condition of strong equicontinuity. Let U be a relatively compact non-trivial open subset of Z, and $\mathcal{V}$ a family of open subsets which cover U as in Proposition 2.1. Let V be an element of $\mathcal{V}$ which meets U, which is assumed to be contained in Z_{i_0} for some $i_0 \in I$, and let $D \subset V$ be an open connected subset with compact closure also contained in V. According to Proposition 2.1, if $h \in \mathcal{H}$ is such that $\operatorname{dom} h \subset D$ and $h(D) \cap U \neq \emptyset$, then there exists an element $\tilde{h} \in \mathcal{H}$ which extends h and whose domain contains V. Moreover, as $\mathcal{H}$ is quasi-analytic and D connected, such extension $\tilde{h}$ is unique on D. In particular, such h admits a unique extension to a homeomorphism of $\overline{D}$ onto its image.

Under the present hypothesis, the completion $\overline{\mathcal{H}}$ of $\mathcal{H}$ is a quasi-analytic pseudogroup of transformations of Z whose action on Z has a single orbit. Let $\overline{\mathcal{H}}_D$ be the collection of homeomorphisms $h|_D$ with $h \in \overline{\mathcal{H}}$ an element whose domain contains D. Let $D' \subset D$ be a connected, compact set with non-empty interior, and let

$$\overline{\mathcal{H}}_{DD'} = \left\{ h \in \overline{\mathcal{H}}_D \mid h(D') \cap D' \neq \emptyset \right\} .$$

By the strong equicontinuity of $\overline{\mathcal{H}}$ and Proposition 2.2, the set D' can be chosen so that all the translates $h(D')$, $h \in \overline{\mathcal{H}}_{DD'}$, are contained in a fixed compact subset K of D. Once this choice of D' is made, let $G = \overline{\mathcal{H}}_{DD'}$ be the resulting space.

The space G is endowed with the compact open topology as a subset of $C(D, Z)$. Every element of G is actually defined on V, hence on $\overline{D}$, and so the compact open topology can be described by the supremum metric given by

$$d(g_1, g_2) = \sup_{x\in D} d_{i_0}(g_1(x), g_2(x)) ,$$

where d_{i_0} is the distance function on $Z_{i_0} \subset Z$ as above.

Lemma 4.1. *Endowed this topology, G is a compact space.*

Proof. It has to be shown that any sequence g_n of elements of G has a convergent subsequence. By equicontinuity, g_n may be assumed to be made of elements of $\mathcal{H}$. By Proposition 2.1 and the definition of G, each g_n can be extended to a homeomorphism whose domain contains V. According to Theorem 2.4, the sequence g_n converges uniformly on D to a map $g \in \widetilde{\mathcal{H}}$. It needs to be shown that $g : D \to g(D)$ is a homeomorphism and that it satisfies $g(D') \cap D' \neq \emptyset$.

To verify this last condition, note that, for each n, there exists $x_n \in D'$ such that $g_n(x_n) \in D'$, by the definition of G. Since D' is compact, it may be assumed that $x_n \to x \in D'$, yielding $g_n(x_n) \to g(x) \in D'$ since $g_n \to g$ uniformly on D. Thus, $g(D') \cap D' \neq \emptyset$.

Each $g_n : D \to g_n(D) \subset V$ is a homeomorphism. Thus, by Proposition 2.1, there are maps $h_n \in \mathcal{H}$ defined on V such that $h_n \circ g_n = \mathrm{id}$ on $g_n(D)$. If g fails to be a homeomorphism on D, then there are points $x, y \in D$ with $d_{i_0}(x, y) > 0$ and $g(x) = g(y) = z$. The map g is a homeomorphism around each point of D, as Theorem 2.4 shows. Thus there are disjoint neighborhoods O_x and O_y of x and y, respectively, such that g maps each of them homeomorphically onto a neighborhood W of z. Since the sequences $g_n(x), g_n(y)$ both converge to z, they may be assumed to be contained in W. Furthermore, perhaps further shrinking W, the restrictions $h_n|_W$ form an equicontinuous family, which therefore converges to a map h which inverts g on W. This situation contradicts the fact that $h_n(g_n(x))$ and $h_n(g_n(y))$ do not have the same limit. □

The following lemma is similar to the corresponding one in Kellum [10].

Lemma 4.2. *The space G, endowed with the compact-open topology and the operations just described, is a locally compact local group.*

Proof. Let g_1, g_2 be two elements of G. Then the composition $g_1 \circ g_2$ is defined on D' because $g_1(D') \subset D$. Therefore there exists $h \in \overline{\mathcal{H}}_D$ which extends $g_1 \circ g_2$. By quasi-analyticity of $\overline{\mathcal{H}}$, this extension is unique and thus it defines a map $(g_1, g_2) \mapsto g_1 \cdot g_2$ from $G \times G$ into $\overline{\mathcal{H}}$. If g_1, g_2 are sufficiently close to the identity of D in the compact open topology of $C(D, Z)$, then also $g_1 \cdot g_2 \in G$.

The existence of a unique identity element e for G, as well as the existence of an inverse operation on G, is proved analogously.

Finally, by Corollary 2.7 and the quasi-analyticity of $\overline{\mathcal{H}}$, it is easy to see that the local group multiplication and inverse map are continuous with the compact open topology on G. □

Remarks. The quasi-analyticity for $\overline{\mathcal{H}}$ was used in this proof. Note that it would be needed even to prove, in a similar way, that Γ is a local group. The final section of the paper discusses the necessity of this condition in some more detail.

By Theorem 2.8, we can assume that all elements of G are isometries with respect to d_{i_0}. Then it easily follows that the above distance d on G is left invariant.

The following lemma follows easily; *cf.* [10].

Lemma 4.3. *The map $G \times D' \to D$ defined by $(g, x) \mapsto g(x)$ makes G into a local group of transformations on $D' \subset D$.*

Let $\Gamma = \mathcal{H} \cap G$, which is a finitely generated dense sub-local group of G. The following is a direct consequence of the minimality of $\mathcal{H}$.

Lemma 4.4. *$\mathcal{H}$ is equivalent to the pseudogroup on any non-empty open subset of D' generated by the local action of Γ on $D' \subset Z$.*

Let x_0 be a point in the interior of D', which will remain fixed from now on. Note that, by construction, all elements of G are defined at x_0. Let $\phi : G \to D$ be the orbit map given by $\phi(g) = g(x_0)$. This map is continuous because the action is continuous.

Lemma 4.5. *The image of the orbit map ϕ contains a neighborhood of x_0.*

Proof. $\mathcal{H}$ is minimal by Theorem 2.3, and thus the space Z is locally homogeneous with respect to the pseudogroup $\overline{\mathcal{H}}$ by Theorem 2.4. More precisely, Proposition 2.1 and Theorem 2.4 show that, given $x \in D'$, there exists $h \in \overline{\mathcal{H}}$ with domain $\operatorname{dom} h = D$ such that $h(x_0) = x$. Since both $x, x_0 \in D'$, it follows that $h \in G$. The statement follows immediately from this. □

Let G_0 denote the collection of elements $g \in G$ such that $g(x_0) = x_0$.

Lemma 4.6. *The set G_0 is a compact subgroup of G.*

Proof. First, G_0 is compact because, being the stabilizer of a point, it is a closed subset of G and G is a compact hausdorff space.

Second, it follows from the definitions of G and of its group multiplication that the product of two elements of G_0 is defined and belongs to G_0, and likewise the inverse of every element. More precisely, if $g_1, g_2 \in G_0$, then $g_1 \circ g_2$ is an element of $\overline{\mathcal{H}}$ which fixes x_0, hence $g_1 \circ g_2(D') \cap D' \neq \emptyset$. □

In the special case of the group G_0 which stabilizes x_0 considered here, the equivalence relation $\sim$ on G used to define a representative coset space of G_0 can also be defined as $h \sim g$ if and only if $h(x_0) = g(x_0)$.

Lemma 4.7. *The orbit map $\phi : G \to Z$ induces a map $\psi : G/G_0 \to Z$ which is a homeomorphism of a neighborhood of the identity class in G/G_0 onto a neighborhood of x_0 in Z.*

Proof. This follows directly from the preceding discussion on coset spaces and the finite dimensionality of Z. □

Corollary 4.8. *$\mathcal{H}$ is equivalent to the pseudogroup induced by the canonical local action of some neighborhood of the identity in Γ on some neighborhood of the identity class in G/G_0.*

Proof. This follows from Lemmas 4.4 and 4.7. □

Corollary 4.9. *G/G_0 is finite dimensional.*

Proof. This follows directly from Lemma 4.7 and the finite dimensionality of Z. □

Lemma 4.10. *The group G_0 contains no non-trivial normal sub-local group of G.*

Proof. If $N \subset G$ is a normal sub-local group contained in G_0 and $n \in N$, then for each g in a suitable neighborhood of e in G there is some $n' \in N$ so that

$$n\,\phi(g) = n\,g(x_0) = g\,n'(x_0) = g(x_0) = \phi(g)\ .$$

Thus n acts trivially on a neighborhood of x_0 in D'. This is possible only if $n = e$, because $\overline{\mathcal{H}}$ is quasi-analytic. □

Lemma 4.11. *The local group G is finite dimensional.*

Proof. By Corollary 4.9, G/G_0 is finite dimensional. By Theorem 1.6, there exists a compact normal subgroup (K, U) in ΔG such that $K \subset G_0$ and $G/(K, U)$ is finite dimensional. Lemma 4.10 implies that K is trivial; thus G is finite dimensional because it is locally isomorphic to G/K. □

Finally, since G_0 is a compact subgroup of G, the following finishes the proof of Theorem 3.3 according to Example 3.2 and Corollary 4.8.

Lemma 4.12. *The group G is a local Lie group.*

Proof. This is a local version of [16, Theorem 73]. By Theorem 1.2, it is enough to show that G has no small subgroups. The local group G is finite dimensional and metrizable, so Theorem 1.4 implies that there is a neighborhood U of e in G which decomposes as the direct product of a local Lie group L and a compact zero-dimensional normal subgroup N. Then $P = N \cap G_0$ is a normal subgroup of G_0, and G_0/P is a Lie group because it is a group which is locally isomorphic to the local Lie group $G/(N,U)$ (*cf.* [9, Theorem 36]).

Furthermore, since N is zero-dimensional, so is P. Thus there exists a neighborhood V of e in G_0 which is the direct product of a connected local lie group M and the normal subgroup P. It may be assumed that $V \subset U$. Since M is connected and N is zero-dimensional, it follows that $M \subset L$.

Summarizing, there is a local isomorphism between G and the direct product $L \times N$, which restricts to a local isomorphism of G_0 to $M \times P$. Therefore, there exists a neighborhood of the class G_0 in G/G_0 which is homeomorphic to a neighborhood of the class of the identity in the product $L/M \times N/P$. It follows that a neighborhood of x_0 in Z is homeomorphic to the product of an euclidean ball and an open subspace $T \subset N/P$. Since Z is by assumption locally connected and N/P zero-dimensional, it follows that T is finite, and hence that N/P is a discrete space. So P is an open subset of N and thus there exists a neighborhood W of e in G such that $W \cap P = W \cap N$. By the local approximation of Jacoby (Theorem 1.3), there exists a compact normal subgroup $K \subset W$ such that G/K is a local Lie group. Then G_0 contains $P \cap K$, which is equal to the normal subgroup $N \cap K$ of G because $K \subset W$. Thus, by Lemma 4.10, $N \cap K$ is trivial. On the other hand, $N/(N \cap K)$ is a zero-dimensional lie group, hence $N \cap K$ is open in N. It follows that N is finite, and thus that G is a local lie group. □

5. A description of transitive, compactly generated, strongly equicontinuous and quasi-effective pseudogroups

The following example is slightly more general than Example 3.2.

Example 5.1. Let G be a locally compact, metrizable and separable local group, $G_0 \subset G$ a compact subgroup, and $\Gamma \subset G$ a dense sub-local group. Suppose that there is a left invariant metric on G inducing its topology. This metric can be assumed to be also G_0-right invariant by the compactness of G_0. Then the canonical local action of Γ on some neighborhood of the identity class in G/G_0 induces a transitive strongly equicontinuous and quasi-effective pseudogroup of local transformations of a locally compact polish space. In fact, this is a pseudogroup of local isometries in the sense of [1].

The proof of the following theorem is a straightforward adaptation of the first part of the proof of Theorem 3.3, by using quasi-effectiveness instead of quasi-analyticity.

Theorem 5.2. *Let $\mathcal{H}$ be a transitive, compactly generated, strongly equicontinuous and quasi-effective pseudogroup of local transformations of a locally compact polish space, and suppose that $\overline{\mathcal{H}}$ is also quasi-effective. Then $\mathcal{H}$ is equivalent to a pseudogroup of the type described in Example 5.1.*

The study of compact generation for the pseudogroups of Example 5.1 is very delicate [12]. But those pseudogroups are obviously complete, and it seems that compact generation could be replaced by completeness in Theorem 5.2, obtaining a better result. This would require the generalization of our work [1] to complete strongly equicontinuous pseudogroups.

6. Quasi-analyticity of pseudogroups

The most elusive of the hypotheses of Theorem 3.3 is that concerning the quasi-analyticity of the closure of a quasi-analytic pseudogroup. We do not have an example of a pseudogroup $\mathcal{H}$ as in the main theorem whose closure fails to be quasi-analytic. Thus this section offers some examples and observations relevant to this problem.

It is quite easy to see that the definition of length space has a local version as described in [1, Section 13], and that the two theorems of Section 15 of such paper are also available for local length spaces.

There are many examples of metric spaces which admit actions of pseudogroups of isometries which are not quasi-analytic. The following two are examples of real trees (see Shalen [18]).

Example 6.1. Let $X = \mathbb{R}^2$ be endowed with the metric given by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$

Given a subset F of the real axis there is an isometry f of X given by

$$f(x, y) = \begin{cases} (x, y) & \text{if } x \in F \\ (x, -y) & \text{if } x \notin F \end{cases}$$

This family of isometries f forms a normal subgroup of the group of isometries of X. Thus this group is not quasi-analytic, although X is not locally compact.

Example 6.2. Let $X = \mathbb{R} * \mathbb{R}$ be the free product of two copies of $\mathbb{R}$. Then X has the structure of a real tree, and is a homogeneous space with respect to its group of isometries. It is not quasi-analytic.

Definition 6.3. A local length space X is analytic at a point $x \in X$ if the following holds: if γ, γ' are geodesic arcs (parametrized by arc-length) defined on an interval about $0 \in \mathbb{R}$, such that $\gamma(0) = \gamma'(0) = x$ and that $\gamma = \gamma'$ on some interval $(-a, 0]$, then they have the same germ at 0. The space X is analytic if it is analytic at every point.

Example 6.4. A riemannian manifold is an analytic length space. Real trees with many branches, as in the above examples, are not analytic.

In relation to Theorem 3.3, if the local length space Z is known to be analytic at one point and admits a transitive action of a pseudogroup of local isometries, then it is analytic.

Proposition 6.5. *Let $\mathcal{H}$ be a pseudogroup of local isometries of an analytic local length space X. Then $\mathcal{H}$ is quasi-analytic.*

Proof. If $\mathcal{H}$ is not quasi-analytic, then, by definition, there exists an element f of $\mathcal{H}$, an open subset U in $\operatorname{dom}(f)$ such that $f|U = \mathrm{id}$, and a point x_0 in the closure of U such that f is not the identity in any neighborhood of x_0. Therefore, there is a sequence of points x_n converging to x_0 such that $f(x_n) \neq x_n$. If n is sufficiently large, then there is a geodesic arc contained in the domain of f and joining x_n and a point $y \in U$. This geodesic arc is mapped by f to a distinct geodesic arc having the same germ at one of its endpoints. □

The following example shows that the converse is false.

Example 6.6. Let X be the euclidean space $\mathbb{R}^2$ endowed with the metric induced by the supremum norm $\|(x,y)\} = \max\{|x|,|y|\}$. Then X is a length space which is not locally analytic. Indeed, if $f : I \to R$ is a function such that $|f(s) - f(t)| \leq |s - t|$ for all $s, t \in I$, then $t \in I \mapsto (t, f(t)) \in X$ is a geodesic. However, every local isometry is locally equal to a linear isometry, hence the pseudogroup of local isometries is quasi-analytic.

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