Gromov's centralizer theorem

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1. Introduction

A fundamental tool in the study of actions of semisimple Lie groups is provided by invariant geometric structures on the manifold being acted upon. This is because geometric structures can be described in terms of reductions or maps defined on frame bundles, thus allowing the use of the theory of cocycles and algebraic hulls developed by Zimmer (cf. [15]–[20]). Invariant geometric structures are particularly useful for this theory when they are rigid because rigidity essentially means that isometries (whether local or infinitesimal) are determined by their jets of a fixed order. Rigid structures were introduced by Gromov in [5], where the problem of extending infinitesimal automorphisms of such structures to local ones was also discussed. In conjunction with results that ensure the extension of local automorphisms to global ones (see, for example, Amores [1]), this establishes an important relation between Zimmer's cocycle machinery and the global features of the manifold on which the action takes place. A remarkable consequence is that whenever a simple group acts on a manifold preserving a finite measure and a suitable rigid geometric structure, the universal cover of the manifold has a space of Killing fields centralizing the action and generating orbits at least as large as those of the group. This result, which is called 'Gromov's centralizer theorem,' is essentially contained in [5] for compact analytic manifolds and provides a useful representation for the fundamental groups of manifolds where simple groups act and is the main tool to prove that such actions are topologically engaging (cf. [20],

* Research supported by N.S.F. Grants DMS-9973086 and DMS-0049077
† Research supported by CONACYT Grant 32197-E

[2]). A fairly detailed, though not complete, exposition of the proof of Gromov's centralizer theorem for compact manifolds is given by Feres [4].

The aim of this work is to give a complete proof of Gromov's centralizer theorem and an extension of it to finite volume manifolds which are not necessarily compact. Our approach is closely related to that of [5] since we establish extension results for automorphisms from infinitesimal to local. However, we differ from [5] in that instead of dealing with automorphisms (i.e., groups) we consider Killing fields (i.e., Lie algebras). This brings us to a linear setup that highly simplifies the task. The nonlinear approach of [5] requires to apply Frobenius' theorem on a space of jets of diffeomorphisms. In our case it is possible to use the results of Nomizu in [11] concerning Killing generators on Riemannian manifolds to complete the infinitesimal-to-local step for Killing fields. This is achieved by restricting the geometric structure to be either a connection or a parallelism on a frame bundle or an order 1 reduction of finite type. Our restriction is not so strong since we show in Propositions 8.4 and 8.5 that under mild conditions (e.g., ergodicity) any geometric structure is essentially a reduction, and we prove in Proposition 7.12 that, for an order 1 reduction, being rigid is equivalent to being of finite type.

It is very plausible that rigidity is equivalent to an appropriate notion of reduction of finite type for higher order frame bundles. A remark of this sort is claimed to be true in [3] and [4], but no actual precise formulation or proof of such fact is known to us. Even though we do not pursue this matter here, we note that simply projecting a high order geometric structure to an order 1 geometric structure does not provide such appropriate setup, as the case of connections shows.

The organization of this article is as follows. In Section 2 we discuss some of the basic properties of Lie groups that are used to define and study geometric structures. In Section 3 we collect some basic results from the theory of analytic manifolds that will be needed in the proof of Gromov's theorem. We also prove an extension of a result originally found in [17] stating that suitable measure preserving actions are locally free almost everywhere, we further note that local freeness holds in the complement of an analytic set. The basic definitions and constructions pertaining to geometric structures are introduced in Section 4. Rigid geometric structures, as originally defined by Gromov in [5], are discussed in Section 5. We further introduce the notion of Killing rigidity (based on Killing vector fields instead of isometries) since our arguments focus on the use of Killing fields; furthermore, we prove that rigidity and Killing rigidity are essentially equivalent. Sections 6 and 7 provide the extension theorems, from infinitesimal to local, for Killing fields associated to parallelisms, connections and structures of finite type. The properties of the algebraic hull of an action that we need are stated in Section 8. The proof of Gromov's centralizer theorem is then completed in Section 9.

Since Gromov's theorem is such a fundamental tool in the study of actions of simple Lie groups, our work allows us to easily generalize several known results. In this regard, in Section 10 we state several applications to actions of simple Lie groups preserving a finite measure on a manifold which is not necessarily compact. The corresponding results for compact manifolds are already known and most of them appear in [20]. Their proofs make use of several techniques like Zimmer's cocycle superrigidity, the properties of topologically engaging actions and Ratner's theorem. However, a common ingredient of such proofs is Gromov's centralizer theorem, in most cases through the use of the fact that such result implies that certain actions are topologically engaging. Since Gromov's theorem was previously known only for compact manifolds, compactness was an essential hypothesis for such results. However, our version of Gromov's theorem replaces compactness by the presence of a finite invariant Zariski measure (e.g., a smooth measure) so one can expect these results to hold for the larger family of finite volume manifolds. It turns out that with the use of our Gromov's centralizer theorem for finite invariant Zariski measures the arguments used to prove the theorems in Section 10 for compact manifolds essentially carry over to the finite volume case. Hence in Section 10 we state our results together with references to the original articles where they appear in the compact case and simply observe that the proofs from such references remain essentially unchanged except for the few remarks that we add.

2. Preliminaries on jet bundles

Let Q be a smooth manifold. If $f : \mathbb{R}^n \to Q$ is a smooth map, then $j^r(f)$ denotes the r-jet of f at the origin $0 \in \mathbb{R}^n$. Let $J^r_n(Q)$ denote the set of r-jets $j^r(f)$ of smooth maps $f : \mathbb{R}^n \to Q$.

For euclidean space $\mathbb{R}^m$, the jet space $J^r_n(\mathbb{R}^m) = \Pi^r_{i=0} S_i(\mathbb{R}^n; \mathbb{R}^m)$, where $S_i(\mathbb{R}^n; \mathbb{R}^m)$ is the vector space of symmetric $\mathbb{R}^m$-valued i-multilinear transformations on $\mathbb{R}^n$.

There is a natural smooth manifold structure on $J^r_n(Q)$ which is defined as follows: if (U, ϕ) is a coordinate chart for Q, where U is an open subset of Q and $\phi : U \to \mathbb{R}^m$ is a diffeomorphism onto an open subset of $\mathbb{R}^m$, then $(J^r_n(U), \widetilde{\phi})$ is a coordinate chart for $J^r_n(Q)$, where $\widetilde{\phi} : J^r_n(U) \to J^r_n(\mathbb{R}^m)$ is the bijection canonically induced by ϕ.

These constructions can be performed in the category of analytic manifolds and analytic maps.

Note also that if Q is a Lie group, then $J^r_n(Q)$ inherits a group structure defined by $j^r(g_1)j^r(g_2) = j^r(g_1 g_2)$.

Let $\mathrm{Gl}^{(k)}(n)$ denote the group of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ that fix 0. As a manifold, $\mathrm{Gl}^{(k)}(n) = \{(A, L_2, \ldots, L_k) \mid A \in \mathrm{Gl}(n), L_i \in S_i(\mathbb{R}^n; \mathbb{R}^n)$ for every $i \geq 2\}$, and is in fact a Lie group. Note that $\mathrm{Gl}^{(1)}(n)$ is the general linear group $\mathrm{Gl}(n)$ and for any pair of integers $l \geq k$ there is a canonical homomorphism $\pi^l_k \colon \mathrm{Gl}^{(l)}(n) \to \mathrm{Gl}^{(k)}(n)$. The kernel of π^k_{k-1} will be denoted by N^k.

Let $\mathfrak{gl}^{(k)}(n)$ denote the space of k-jets at 0 of vector fields on $\mathbb{R}^n$ that vanish at 0. The bracket of two elements $j^k(X), j^k(Y) \in \mathfrak{gl}^{(k)}(n)$ is defined by

$$[j^k(X), j^k(Y)]^k = -j^k([X, Y]).$$

A direct computation shows that, since both X and Y vanish at 0, the k-jet of $[X, Y]$ depends only on the k-jets of X and Y, and so $[\ ,\ ]^k$ defines a Lie algebra structure on $\mathfrak{gl}^{(k)}(n)$. Observe that $\mathfrak{gl}^{(1)}(n)$ can be naturally identified with the space of $n \times n$ real matrices and that, with the above bracket operation, it is canonically isomorphic to the general linear Lie algebra $\mathfrak{gl}(n)$.

The following lemma provides natural representations for $\mathrm{Gl}^{(k)}(n)$ and $\mathfrak{gl}^{(k)}(n)$, and shows that $\mathfrak{gl}^{(k)}(n)$ is the Lie algebra of $\mathrm{Gl}^{(k)}(n)$.

LEMMA 2.1. *Let V be the space of $(k-1)$-jets at 0 of vector fields defined in a neighborhood of 0 in $\mathbb{R}^n$. Let $H \colon \mathrm{Gl}^{(k)}(n) \to \mathrm{Gl}(V)$ and $h \colon \mathfrak{gl}^{(k)}(n) \to \mathfrak{gl}(V)$ be defined by*

$$\begin{aligned} H(j^k(\varphi))(j^{k-1}(Z)) &= j^{k-1}(d\varphi(Z)) \\ h(j^k(X))(j^{k-1}(Z)) &= -j^{k-1}(L_X(Z)), \end{aligned}$$

for all $j^{k-1}(Z) \in V$. Then both H and h are faithful representations, and with respect to them $\mathfrak{gl}^{(k)}(n)$ is the Lie algebra of $\mathrm{Gl}^{(k)}(n)$. Moreover, the exponential map $\exp \colon \mathfrak{gl}^{(k)}(n) \to \mathrm{Gl}^{(k)}(n)$ *is given by*

$$\exp(t j^k(X)) = j^k(d\varphi_t),$$

for $j^k(X) \in \mathfrak{gl}^{(k)}(n)$ and $t \in \mathbb{R}$, where φ_t is the local flow associated to X.

Proof. A direct computation shows that H and h are injective smooth homomorphisms. Hence the last claim, with 'exp' being the exponential map of $\mathrm{Gl}(V)$, implies that $\mathfrak{gl}^{(k)}(n)$ is the Lie algebra of $\mathrm{Gl}^{(k)}(n)$. It then suffices to verify that the exponential map has the expression stated above.

If $j^k(X) \in \mathfrak{gl}^{(k)}(n)$, then X vanishes at 0 and so its local flow φ_t is defined at 0 for every $t \in \mathbb{R}$. In particular, the domain of the local flow φ_t contains $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^n$, and since such domain is open it follows that the map φ_t is defined in a neighborhood of 0, for every $t \in \mathbb{R}$. On the other hand, X and φ_t are related by (cf. [7])

$$-L_X Z = \left.\frac{d}{dt}\right|_{t=0} d\varphi_t(Z).$$

Thus it follows from the definitions of H and h that the map $\mathbb{R} \to \mathrm{Gl}(V)$ defined by $t \mapsto j^k(d\varphi_t)$ is a one-parameter subgroup of $\mathrm{Gl}^{(k)}(n)$ whose velocity vector at $t = 0$ is $j^k(X)$, and hence that $\exp(tj^k(X)) = j^k(d\varphi_t)$ for every $t \in \mathbb{R}$.

The above representations are isomorphisms when $k = 1$. Moreover, the homomorphism H realizes $\mathrm{Gl}^{(k)}(n)$ as a real algebraic subgroup of $\mathrm{Gl}(V)$.

Let $a : \mathrm{Gl}^{(k+r)}(n) \to J^r_n(\mathrm{Gl}^{(k)}(n))$ be defined as follows. If $g \in \mathrm{Gl}^{(k+r)}(n)$ is of the form $g = j^{(k+r)}(f)$, let $f_k : \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$ be the map given by

$$f_k(x) = j^k(\tau_{-x} \circ f \circ \tau_{f^{-1}(x)}),$$

where $\tau_v(y) = y + v$ is the translation by v in $\mathbb{R}^n$, and set

$$a(g) = j^r(f_k).$$

This map a satisfies

$$a(g_1 g_2) = a(g_1)(a(g_2) \circ \pi^{k+r}_r(g_1^{-1})),$$

where π^{k+r}_r is the natural projection of $(k + r)$-jets into r-jets and where '$\circ$' denotes the operation given by $j^r(f) \circ j^r(\varphi) = j^r(f \circ \varphi)$. Thus, if $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ is the semi-direct product with group multiplication $(g, h)(g', h') = (gg', h(h' \circ g^{-1}))$, then the map $(\pi^{k+r}_r, a) : \mathrm{Gl}^{(k+r)}(n) \to \mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ is a homomorphism of Lie groups.

LEMMA 2.2. *The homomorphism* $(\pi^{k+r}_r, a) : \mathrm{Gl}^{(k+r)}(n) \to \mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ *maps* N^{k+r} *into* $J^r_n(N^k)$.

Proof. Each $g \in N^{k+r}$ can be represented as $g = j^{k+r}(\varphi)$, where $\varphi(y) = y + (1/(k + r)!)L(y, \ldots, y)$, L being a symmetric $\mathbb{R}^n$-valued $(k + r)$-multilinear map on $\mathbb{R}^n$. Correspondingly, $a(g) = j^r(f)$, where $f : \mathbb{R}^n \to N^k \subset \mathrm{Gl}^{(k)}(n)$ is given by $f(x) = (I, 0, \ldots, 0, (1/r!)L_{x^{(r)}})$, and $L_{x^{(r)}}$ is the $\mathbb{R}^n$-valued k-multilinear map defined by $(y_1, \ldots, y_k) \mapsto L(y_1, \ldots, y_k, x, \ldots, x)$.

If Q is a manifold that admits a smooth (respectively, analytic) action of $\mathrm{Gl}^{(k)}(n)$ (on the left), then $J^r_n(Q)$ admits a smooth (respectively, analytic) action of $\mathrm{Gl}^{(k+r)}(n)$ (on the left) which is called the r-prolongation of the action on Q.

This r-prolongation is constructed as follows. There is a natural left action of the Lie group $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ on $J^r_n(Q)$ given by $(g,h)q = h \cdot (q \circ g^{-1})$, where '$\circ$' is defined as before and '$\cdot$' is given by $j^r(f_1) \cdot j^r(f_2) = j^r(f_1 f_2)$. This induces, with the help of (π^{k+r}_r, a), a canonical action $\mathrm{Gl}^{(k+r)}(n) \times J^r_n(Q) \to J^r_n(Q)$ given by $gq = a(g) \cdot (q \circ \pi^{k+r}_r(g^{-1}))$.

Finally, there is a natural smooth map $j \colon J^{r+1}_n(Q) \to J^1_n(J^r_n(Q))$ given by $j^{r+1}(f) \mapsto j^1(f_r)$, where $f_r(x) = j^r(f \circ \tau_x)$ for $f \colon \mathbb{R}^n \to Q$. The above discussion defines an action of $\mathrm{Gl}^{(k+r+1)}(n)$ on $J^{r+1}_n(Q)$, and this same construction applied to the action of $\mathrm{Gl}^{(k+r)}(n)$ on $J^r_n(Q)$ defines an action of $\mathrm{Gl}^{(k+r+1)}(n)$ on $J^1_n(J^r_n(Q))$. It is straightforward to show from the definitions that j is $\mathrm{Gl}^{(k+r+1)}(n)$-equivariant with respect to such actions. In words, the $(r+1)$-prolongation of the action of $\mathrm{Gl}^{(k)}(n)$ on Q and the 1-prolongation of the action of $\mathrm{Gl}^{(k+r)}(n)$ on $J^r_n(Q)$ coincide on the subset $J^{r+1}_n(Q)$ of $J^1_n(J^r_n(Q))$.

If a group G acts on a manifold M and $X \in \mathfrak{g}$, the Lie algebra of G, the vector field induced by the one-parameter group of diffeomorphisms $t \mapsto \exp(tX)$ will be denoted by X^*.

3. Analytic manifolds and actions

This section contains some results and remarks from the real analytic category that will be needed later. Further details, as well as proofs, can be found in Narasimhan [10].

DEFINITION 3.1. A subset W of an analytic manifold M is called an analytic set if for each $x \in W$ there exists an open neighborhood U of x in M and an analytic function f defined on U such that $W \cap U = f^{-1}(0)$.

The well-known property that an analytic function on a connected set is determined by any of its germs provides the following result.

PROPOSITION 3.2. *The only analytic set with nonempty interior in a connected analytic manifold M is M itself.*

Proof. It suffices to show that if W is analytic in M and has nonempty interior, then W is also open. Fix an interior point x_0 of W and let x be any given point of W. Since M is connected, there is a finite sequence $U_0, \ldots, U_k$ of connected open subsets of M so that $x_0 \in U_0$, $x \in U_k$,

$U_{i-1} \cap U_i \neq \emptyset$ for every $i = 1, \ldots, k$ and so that each $W \cap U_i$ is the zero set of finitely many analytic functions defined on U_i. Since x_0 is an interior point of W, the analytic functions on U_0 whose common zero set equals $W \cap U_0$ vanish on an open nonempty set, so they are null on all of U_0 and thus $U_0 \subset W$. Assuming that $U_i \subset W$, and since $U_i \cap U_{i+1} \neq \emptyset$, the same argument implies that $U_{i+1} \subset W$. After a finite number of steps we conclude that $U_k \subset W$, and so x is an interior point of W.

DEFINITION 3.3. Let W be an analytic subset of an analytic manifold M. A point $x \in W$ is called regular of dimension p if there is an open neighborhood U of x in M so that $U \cap W$ is an analytic submanifold of M of dimension p. A point is called singular if it is not regular.

According to [10], the usual definition of C-analytic set is equivalent to the following one.

DEFINITION 3.4. An analytic subset W of an open subset Ω of $\mathbb{R}^n$ is called C-analytic if there are finitely many analytic functions $f_1, \ldots, f_k$ defined on Ω so that $W = \bigcap_{i=1}^{k} f_i^{-1}(\{0\})$.

With respect to a suitable analytic coordinate system, an analytic variety W in an analytic manifold M is locally a C-analytic set. More precisely, for every $x \in M$ there is an analytic diffeomorphism $\phi: U \to \Omega$, where U is open in M and Ω is open in $\mathbb{R}^n$ so that $\phi(W \cap U)$ is C-analytic. This simple fact implies several important local properties for analytic sets on analytic manifolds. In particular, the following result follows from the study of C-analytic sets developed in [10].

PROPOSITION 3.5. *Let W be an analytic subset of an analytic manifold M. Then there is a countable collection $(B_i)_i$ of analytic submanifolds of M which cover W and such that for every i the dimension of B_i is less than or equal to the maximum of the dimensions of the regular points of W.*

This proposition and the previous discussion have the following corollary as a direct consequence.

COROLLARY 3.6. *Any proper analytic subset W of a connected analytic manifold has empty interior, and thus its complement is dense. Moreover W is null with respect to the smooth measure class.*

DEFINITION 3.7. A measure on an analytic manifold M is called a Zariski measure if every proper analytic variety of a connected open subset of M is a null set.

Observe that analytic subsets of open subsets of an analytic manifold are Borel sets because they are locally closed.

By Corollary 3.6, a smooth measure on an analytic manifold M is a Zariski measure. Nontrivial examples of finite Zariski measures which are not smooth are the so called Patterson-Sullivan measures supported on limit sets of discrete groups of isometries of hyperbolic spaces acting on the sphere at infinity, see [12, 14]. These measures are in fact quasi-invariant under an analytic action of a discrete group.

Let G be a connected, simple Lie group acting smoothly on a manifold M. It has been proved by Zimmer [17] that if the action is not trivial and leaves invariant a finite smooth measure, then it is locally free on a conull, open, dense subset of M. (An action is called locally free if its stabilizers are discrete.) The following result gives further information on this conull set when the manifold and the action are analytic.

PROPOSITION 3.8. *Let M be a connected analytic manifold, and let G be a connected, noncompact, simple Lie group which acts on M analytically and preserving a finite Zariski measure. If the action is nontrivial, then there is a G-invariant proper analytic set W of M such that the action of G is locally free on $M \setminus W$.*

Proof. Let V be the real algebraic variety consisting of the subspaces of the Lie algebra of G. The map $\phi: M \to V$ which assigns to $x \in M$ the Lie algebra $\phi(x) = \mathfrak{g}_x$ of the stabilizer G_x of x is measurable and G-equivariant, where G acts on V via the adjoint representation. If μ is the finite measure preserved by G, then $\phi_*(\mu)$ is an $\mathrm{Ad}(G)$-invariant measure on V. Borel's density theorem (cf. Zimmer [15]) then implies that $\phi_*(\mu)$ is supported on fixed points of the action of $\mathrm{Ad}(G)$. From this it follows that, for almost every point $x \in M$ with respect to the measure μ, the subspace $\mathfrak{g}_x$ is an ideal. Since G is simple and connected, it must be that, for almost every $x \in M$, the stabilizer G_x is either G or a discrete subgroup of G.

Let $X_1, \ldots, X_k$ be a basis of the Lie algebra of G. For $x \in M$, let $r(x)$ denote the dimension of the subspace of T_xM generated by the vector fields $X_1^*, \ldots, X_k^*$ at x. Let r be the maximum of $r(x)$ as x varies on M, and let W be set of points $x \in M$ where $r(x) \leq r - 1$. Using local analytic coordinates and the fact that the condition $r(x) \leq r-1$ can be expressed in terms of the vanishing of suitable determinants, it obtains that W is an analytic subset of M. Moreover, W is proper, because there is some $x \in M$ such that $r(x) = r$. Thus, W is null with respect to μ.

If $r < k$, then it follows from the first paragraph of this proof that the stabilizer $G_x = G$, for almost every $x \in M$, and that there is a point $x \in M \setminus W$ such that $G_x = G$. It thus follows, by the definitions of r and W, that $r = 0$ and so that the action of G is trivial. It must then be that $r = k$, and hence that the action of G is locally free on the nonempty open set $M \setminus W$, where W is a proper analytic subset of M. The last claim follows from Corollary 3.6 and the definition of Zariski measure.

4. Geometric structures

In this section, and in the sections that follow, M will denote a smooth manifold of dimension n. Some results require the hypothesis that M be analytic. Such hypothesis will be explicitely made when needed.

Let $L^{(k)}(M)$ denote the k-th order frame bundle of M. This is the principal fiber bundle over M whose elements are the k-jets at the origin $0 \in \mathbb{R}^n$ of diffeomorphisms from a neighborhood of 0 in $\mathbb{R}^n$ into M. The structure group of $L^{(k)}(M)$ is $\mathrm{Gl}^{(k)}(n)$, with a natural (right) action given by composition of diffeomorphisms. Geometric structures on M arise as sections of bundles associated to $L^{(k)}(M)$.

Let $T_x^{(k)}M$ be the vector space of $(k-1)$-jets of vector fields at $x \in M$. Then the set $T^{(k)}M = \bigcup_{x \in M} T_x^{(k)}M$ has the structure of smooth vector bundle over M. Also, the map

$$\begin{aligned} L^{(k)}(M) \times T_0^{(k)}\mathbb{R}^n &\to \qquad T^{(k)}M \\ (j_0^k(\varphi), j_0^{k-1}(X)) &\mapsto \; j_x^{k-1}(d\varphi(X)) \end{aligned}$$

makes $T^{(k)}M$ into a vector bundle associated to $L^{(k)}(M)$.

DEFINITION 4.1. Let Q be a manifold on which $\mathrm{Gl}^{(k)}(n)$ acts smoothly on the left, and let $Q^k(M)$ be the fiber bundle over M associated to $L^{(k)}(M)$ and such action (that is, $Q^k(M)$ is the quotient of $L^{(k)}(M) \times Q$ by the action of $\mathrm{Gl}^{(k)}(n)$ given by $(\alpha, q)g = (\alpha g, g^{-1}q)$). A geometric structure of order k and type Q on M is a smooth section of $Q^k(M)$.

There is a natural correspondence between sections of $Q^k(M)$ and $\mathrm{Gl}^{(k)}(n)$-equivariant maps $L^{(k)}(M) \to Q$. This second interpretation will often be used when referring to a geometric structure.

A geometric structure on M will be called of algebraic type Q (or simply of algebraic type) if Q is a real algebraic variety and the action of $\mathrm{Gl}^{(k)}(n)$ is algebraic. If H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$, then an H-structure of order k is a reduction of $L^{(k)}(M)$ to the group H. If H is closed in $\mathrm{Gl}^{(k)}(n)$, then there is a natural correspondence between the H-structures of order k and the geometric structures of order k and type $\mathrm{Gl}^{(k)}(n)/H$. An H-structure of order k will be called of algebraic type if H is a real algebraic subgroup of $\mathrm{Gl}^{(k)}(n)$. Hence an H-structure of order k and algebraic type defines a geometric structures of order k and type $\mathrm{Gl}^{(k)}(n)/H$ which is of algebraic type as well.

Note that if H is not a closed subgroup, then an H-structure does not necessarily provide the section that defines a geometric structure as given in Definition 4.1. Since most of the properties for geometric structures will be given using the section that defines it or the corresponding equivariant map on a frame bundle, some of our definitions do not apply to arbitrary H-structures. For the latter, we can provide alternative definitions for most basic concepts, but we will not do so since our main results deal with structures of algebraic type. Hence, when considering geometric notions associated to an H-structure we will sometimes restrict to closed subgroups H, but we will be very explicit when doing this.

Associated to a geometric structure of order k and type Q on M and to a non-negative integer r, there is a structure of order $k+r$ and type $J^r_n(Q)$, which is called the r-th order prolongation of the structure. If $\sigma : L^{(k)}(M) \to Q$ is the $\mathrm{Gl}^{(k)}(n)$-equivariant map that defines a geometric structure on M, then the r-th order prolongation of σ is the geometric structure whose associated $\mathrm{Gl}^{(k+r)}(n)$-equivariant map is given by

$$\begin{aligned} \sigma^r : L^{(k+r)}(M) &\to J^r_n(Q) \\ j^{k+r}(\varphi) &\mapsto j^r(\sigma(j^k(\varphi \circ \tau_\bullet))) \end{aligned}$$

where $\sigma(j^k(\varphi \circ \tau_\bullet))$ denotes the map $v \in \mathbb{R}^n \mapsto \sigma(j^k(\varphi \circ \tau_v)) \in Q$. Note that $\sigma^0 = \sigma$ for any geometric structure σ.

A smooth diffeomorphism f of M induces a bundle diffeomorphism $f_{(k)}$ of $L^{(k)}(M)$. Through such maps, the group of smooth diffeomorphisms of M naturally acts on the associated bundles $Q^k(M)$ described above and on their sections. If σ is such a section (i.e., a geometric structure on M), then the group of diffeomorphisms of M that preserve σ is denoted by $\mathrm{Aut}(\sigma)$ and is called the group of automorphisms (or isometries) of σ. Similarly, $\mathrm{Aut}^{\mathrm{loc}}(\sigma)$ denotes the pseudogroup of local diffeomorphisms of M which preserve σ.

For $x, y \in M$, let $D^{(k)}_{x,y}(M)$ denote the space of k-jets of diffeomorphisms from a neighborhood of x in M into M and which send x to y. Let $D^{(k)}_x(M) = D^{(k)}_{x,x}(M)$, the group of k-jets of diffeomorphisms of M which fix x. The group $D^{(k)}_x(M)$ has a Lie group structure under which it is isomorphic to $\mathrm{Gl}^{(k)}(n)$. Its Lie algebra is $\mathcal{D}^{(k)}_x(M)$, the space of k-jets at x of vector fields on M vanishing at x (with Lie algebra structure similar to that of $\mathfrak{gl}^{(k)}(n)$ described in Section 2). Moreover, Lemma 2.1 remains valid if $D_x^{(k)}(M)$, $\mathcal{D}_x^{(k)}(M)$, and $T_x^{(k)}M$ replace $\mathrm{Gl}^{(k)}(n)$, $\mathfrak{gl}^{(k)}(n)$, and V, respectively.

If $j_x^k(f) \in D_{x,y}^{(k)}(M)$, then the action of $f_{(k)}$ on $L^{(k)}(M)$ maps the fiber of $L^{(k)}(M)$ over x onto the fiber over y so that the $\mathrm{Gl}^{(k)}(n)$-equivariant map which it defines on such fibers depends only on the jet $j_x^k(f)$. In particular, the Lie group $D_x^{(k)}(M)$ acts on the fiber of $L^{(k)}(M)$ over x, and this action commutes with the action of $\mathrm{Gl}^{(k)}(n)$.

DEFINITION 4.2. Let σ be a geometric structure of order k and type Q on M. For $x, y \in M$ the set $\mathrm{Aut}^{k+r}(\sigma, x, y)$ of infinitesimal automorphisms of σ of order $k+r$ taking x to y consists of the elements $g \in D_{x,y}^{(k+r)}(M)$ which preserve σ up to order r, i.e., that satisfy $\sigma^r(g\alpha) = \sigma^r(\alpha)$ for every α in the fiber of $L^{(k+r)}(M)$ above x, where σ^r denotes the r-th prolongation of σ considered as a $\mathrm{Gl}^{(k+r)}(n)$-equivariant map over $L^{(k+r)}(M)$.

Let $\mathrm{Aut}^{k+r}(\sigma, x)$ denote $\mathrm{Aut}^{k+r}(\sigma, x, x)$. This is a closed subgroup of $D_x^{(k+r)}(M)$ and hence a Lie subgroup. Note that for an element $g \in D_{x,y}^{(k+r)}(M)$ to belong to $\mathrm{Aut}^{k+r}(\sigma, x, y)$ it is enough to have $\sigma^r(g\alpha) = \sigma^r(\alpha)$, for some $\alpha \in L^{(k+r)}(M)$. This follows from the $\mathrm{Gl}^{(k+r)}(n)$-equivariance of σ^r.

LEMMA 4.3. *Let $\sigma\colon L^{(k)}(M) \to Q$ be a geometric structure on M. Then for every $x \in M$ and every α in the fiber of $L^{(k+r)}(M)$ over x, the group $\mathrm{Aut}^{k+r}(\sigma, x)$ is isomorphic to the stabilizer of $\sigma^r(\alpha)$ under the action of $\mathrm{Gl}^{(k+r)}(n)$ on $J_n^r(Q)$.*

Proof. Let $\alpha \in L^{(k+r)}(M)$ be an element in the fiber over x and let $g \in D_x^{(k+r)}(M)$. Then $\alpha^{-1} \circ g^{-1} \circ \alpha \in \mathrm{Gl}^{(k+r)}(n)$, and the properties described above give the identities

$$\sigma^r(g\alpha) = \sigma^r((\alpha \circ \alpha^{-1}) \circ g\alpha) = \sigma^r(\alpha(\alpha^{-1} \circ g \circ \alpha)) = (\alpha^{-1} \circ g^{-1} \circ \alpha)\sigma^r(\alpha),$$

where '$\circ$' denotes composition of the diffeomorphisms representing the jets paired by '$\circ$.' It follows that $g \in \mathrm{Aut}^{k+r}(\sigma, x)$ if and only if $\alpha^{-1} \circ g^{-1} \circ \alpha$ stabilizes $\sigma^r(\alpha)$, from what it obtains that the map $g \mapsto \alpha^{-1} \circ g \circ \alpha$ provides the required isomorphism.

A corollary to this result is that if σ is a geometric structure of algebraic type, then $\mathrm{Aut}^{k+r}(\sigma, x)$ is a real algebraic subgroup of $D_x^{(k+r)}(M)$, where the latter has an obvious structure of real algebraic group isomorphic to $\mathrm{Gl}^{(k+r)}(n)$.

If X is a vector field on M, its local flow lifts to a local flow on the bundle $L^{(k)}(M)$, and thus it defines a vector field $X_{(k)}$ on $L^{(k)}(M)$ which is called the lift of X to $L^{(k)}(M)$. Such vector fields are characterized by the properties described in the following lemma (cf. [6], [7]).

LEMMA 4.4. *Let X be a vector field on M. Then for every k, the vector field $X_{(k)}$ is the unique vector field on $L^{(k)}(M)$ that projects to X and such that:*

1. $[X_{(k)}, Y^*] = 0$ *for every* $Y \in \mathfrak{gl}^{(k)}(n)$,
2. $L_{X_{(k)}}\theta_k = 0$,

where θ_k is the canonical form of $L^{(k)}(M)$. Conversely, if Z is a vector field on $L^{(k)}(M)$ that satisfies the above conditions, then there is a unique vector field X on M such that $X_{(k)} = Z$.

A vector field Z as in the previous lemma will be called projectable. As a consequence of this we have the following.

LEMMA 4.5. *Let P be a smooth reduction of $L(M)$ to a group with Lie algebra $\mathfrak{h}$. Let Z be a smooth vector field over P that satisfies:*

1. $[Z, Y^*] = 0$ *for every* $Y \in \mathfrak{h}$,
2. $L_Z\theta = 0$,

where θ is the canonical form of P. Then there is a smooth vector field X on M such that $Z = X_{(1)}|_P$.

We observe that the previous two results hold in the analytic category as well.

DEFINITION 4.6. A Killing field for a geometric structure σ on M is a vector field on M whose local flow acts on M by local automorphisms of σ. The space of Killing fields and local Killing fields of a geometric structure σ are denoted by $\mathrm{Kill}(\sigma)$ and $\mathrm{Kill}^{\mathrm{loc}}(\sigma)$, respectively.

Let σ be a geometric structure on M, viewed as a map defined on $L^{(k)}(M)$. It follows from the standard relation between a vector field and its local flow that a vector field X on M is a Killing field if and only if $d\sigma_\alpha(X_{(k)}) = 0$ for every $\alpha \in L^{(k)}(M)$. This suggests the following definition.

DEFINITION 4.7. Let σ be a geometric structure of order k on M. An infinitesimal Killing field of order $k+r$ at x for σ is a $(k+r)$-jet $j_x^{k+r}(X)$ of a germ at x of a vector field X so that $d\sigma_\alpha^r(X_{(k)}) = 0$ for every $\alpha \in L^{(k+r)}(M)$ that lies in the fiber over x. Let $\mathrm{Kill}^{k+r}(\sigma, x)$ denote the space of infinitesimal Killing fields for σ of order $k+r$, and let $\mathrm{Kill}_0^{k+r}(\sigma, x)$ denote the subspace consisting of those vanishing at x.

The above definition does not depend on the choice of the vector field because for a given vector field X the value of $d\sigma_\alpha^r(X_{(k)})$ depends only on $j_x^{k+r}(X)$.

Let $j_x^{k+r}(X) \in \mathcal{D}_x^{(k+r)}(M)$. Then a straightforward computation shows that $j_x^{k+r}(X) \in \mathrm{Kill}^{k+r}(\sigma, x)$ if and only if, for φ_t the local of X, we have $j_x^{k+r}(d\varphi_t) \in \mathrm{Aut}^{k+r}(\sigma, x, \varphi_t(x))$, for every t in a neighborhood of 0. Moreover, for $j_x^{k+r}(X) \in \mathrm{Kill}_0^{k+r}(\sigma, x)$ we have the latter condition for every $t \in \mathbb{R}$. From this it follows that $\mathrm{Kill}_0^{k+r}(\sigma, x)$ is the Lie algebra of $\mathrm{Aut}^{k+r}(\sigma, x)$. Moreover, the exponential map of $\mathrm{Aut}^{k+r}(\sigma, x)$ is given by an expression similar to that in Lemma 2.1.

5. Rigid structures

In this section, we recall the concept of rigid geometric structure, originally defined by Gromov [5], and its basic properties. We also introduce the related notion of Killing rigidity for geometric structures.

DEFINITION 5.1. Let r be a non-negative integer. A geometric structure σ of order k on M is said to be r-rigid if, for every $x \in M$, the canonical projection $\pi_{k+r}^{k+r+1}\colon \mathrm{Aut}^{k+r+1}(\sigma, x) \to \mathrm{Aut}^{k+r}(\sigma, x)$ is injective. The geometric structure σ is called Killing r-rigid if for every $x \in M$ the canonical projection $\pi_{k+r}^{k+r+1}\colon \mathrm{Kill}_0^{k+r+1}(\sigma, x) \to \mathrm{Kill}_0^{k+r}(\sigma, x)$ is injective.

It is easy to verify that σ is Killing r-rigid if and only if, for every $x \in M$, the map $\pi_{k+r}^{k+r+1}\colon \mathrm{Kill}^{k+r+1}(\sigma, x) \to \mathrm{Kill}^{k+r}(\sigma, x)$ is injective.

As noted in Section 4, $\mathrm{Kill}_0^{k+r+1}(\sigma, x)$ is the Lie algebra of the group $\mathrm{Aut}^{k+r+1}(\sigma, x)$. It follows easily from this that r-rigidity implies Killing r-rigidity, and also that σ is Killing r-rigid if and only if the projection $\pi_{k+r}^{k+r+1}\colon \mathrm{Aut}^{k+r+1}(\sigma, x) \to \mathrm{Aut}^{k+r}(\sigma, x)$ has discrete kernel.

For every $l \geq k$, the group $\mathrm{Aut}^l(\sigma, x)$ acts simply transitively on $\mathrm{Aut}^l(\sigma, x, y)$, in such a way that the fibers of the natural projection $\pi_{k+r}^{k+r+1}\colon \mathrm{Aut}^{k+r+1}(\sigma, x, y) \to \mathrm{Aut}^{k+r}(\sigma, x, y)$ are the orbits of the kernel of $\pi_{k+r}^{k+r+1}\colon \mathrm{Aut}^{k+r+1}(\sigma, x) \to \mathrm{Aut}^{k+r}(\sigma, x)$. From this it follows that σ

is r-rigid if and only if $\pi^{k+r+1}_{k+r}\colon \mathrm{Aut}^{k+r+1}(\sigma,x,y) \to \mathrm{Aut}^{k+r}(\sigma,x,y)$ is injective for every $x,y \in M$, and also that σ is Killing r-rigid if and only if $\pi^{k+r+1}_{k+r}\colon \mathrm{Aut}^{k+r+1}(\sigma,x,y) \to \mathrm{Aut}^{k+r}(\sigma,x,y)$ has discrete fibers for every $x,y \in M$.

The following result provides an alternative characterization of rigidity.

PROPOSITION 5.2. *Let $\sigma : L^{(k)}(M) \to Q$ be the $\mathrm{Gl}^{(k)}(n)$-equivariant map defining a geometric structure of order k and type Q on M. Then σ is r-rigid ($r \geq 0$) if and only if the action of N^{k+r+1} on the image of σ^{r+1} is free. Also, σ is Killing r-rigid if and only if the action of N^{k+r+1} on the image of σ^{r+1} is locally free.*

Proof. Let $\alpha \in L^{(k+r+1)}(M)$. By Lemma 4.3, if S denotes the stabilizer of $\sigma^{r+1}(\alpha)$ under the action of $\mathrm{Gl}^{(k+r+1)}(n)$, then there is an isomorphism $\varphi\colon \mathrm{Aut}^{k+r+1}(\sigma,x) \to S$. Furthermore, the proof of such lemma shows that the kernel of the natural projection $\mathrm{Aut}^{k+r+1}(\sigma,x) \to \mathrm{Aut}^{k+r}(\sigma,x)$ is mapped by φ onto $S \cap N^{k+r+1}$. Both statements follow immediately from this.

LEMMA 5.3. *Suppose that $\mathrm{Gl}^{(k)}(n)$ acts smoothly on a manifold Q. If N^k acts locally freely on Q, then N^{k+1} acts freely on $J^1_n(Q)$.*

Proof. Let $g \in N^{k+1}$ and let $j^1(h) \in J^1_n(Q)$ be such that $gj^1(h) = j^1(h)$. It follows from Lemma 2.2 and its proof that there exists an N^k-valued function f defined on an open neighborhood of 0 in $\mathbb{R}^n$ such that $f(0)$ is the identity element of N^k and $gj^1(h) = j^1(fh)$. If $R(h(0))\colon N^k \to Q$ is the map given by $n \mapsto nh(0)$, then a direct calculation gives

$$dh_0 + dR(h(0))_{f(0)} \circ df_0 = dh_0.$$

The map $R(h(0))$ is an immersion because the action of N^k on Q is locally free. Thus $df_0 = 0$, and so g is the identity element of N^{k+1}.

The next proposition relates the concepts of rigidity and Killing rigidity for geometric structures. It shows that if a geometric structure is r-rigid or Killing r-rigid, then it is l-rigid for every $l \geq r+1$.

PROPOSITION 5.4. *Let σ be a geometric structure of order k and type Q on M. If σ is Killing r-rigid, then it is $(r+1)$-rigid.*

Proof. By Proposition 5.2, the group N^{k+r+1} acts locally freely on the image of σ^{r+1}, and thus, because the action is smooth, it also acts locally freely on an N^{k+r+1}-invariant open subset $Q' \subset J^{r+1}_n(Q)$ that contains the image of σ^{r+1}. Hence, by Lemma 5.3, N^{k+r+2} acts freely on $J^1_n(Q')$. Taking into account the immersion $J^{r+2}_n(Q) \subset J^1_n(J^{r+1}_n(Q))$

discussed in the remarks that follow Lemma 2.2, it obtains that N^{k+r+2} acts freely on $J_n^{r+2}(Q) \cap J_n^1(Q')$. Since the image of σ^{r+1} is contained in the open set Q', the image of σ^{r+2} is contained in $J_n^{r+2}(Q) \cap J_n^1(Q')$. Therefore, N^{k+r+2} acts freely on the image of σ^{r+2}, and thus σ is $(r+1)$-rigid.

COROLLARY 5.5. *Let σ be an r-rigid (respectively, Killing r-rigid) geometric structure of order k on M. If $t \geq s \geq r$ (respectively, $t \geq s \geq r+1$), the natural map $\pi_{k+s}^{k+t} \colon \mathrm{Aut}^{k+t}(\sigma, x, y) \to \mathrm{Aut}^{k+s}(\sigma, x, y)$ is injective, for all $x, y \in M$.*

Moreover, if σ is of algebraic type, then for every $x \in M$ there exists an integer $s(x) \geq r$ such that π_{k+s}^{k+t} is bijective whenever $t \geq s \geq s(x)$.

Proof. When $x = y$ the first part of the proposition follows from the previous result. In the general case, it suffices to consider the simply transitive action of $\mathrm{Aut}^{k+s}(\sigma, x)$ on $\mathrm{Aut}^{k+s}(\sigma, x, y)$.

Moreover, if σ is of algebraic type, then by the proof of Lemma 4.3, the groups $\mathrm{Aut}^{k+s}(\sigma, x)$ are real algebraic subgroups of $D_x^{(k+r)}(M)$, for all $s \geq r$. Using the fact that the inclusions π_{k+s}^{k+t} $(t \geq s)$ are algebraic maps, they give rise to a decreasing sequence of real algebraic groups. By the Noetherian property of algebraic groups, this sequence must stabilize, thus proving the last part of the proposition.

When considering Killing vector fields, the last statement in this corollary does not require that the geometric structure be of algebraic type.

COROLLARY 5.6. *Let σ be a Killing r-rigid geometric structure of order k on M. Then for every $x \in M$, there is an integer $s(x) \geq r$ such that the natural map $\pi_{k+t}^{k+t+1} \colon \mathrm{Kill}^{k+t+1}(\sigma, x) \to \mathrm{Kill}^{k+t}(\sigma, x)$ is an isomorphism for every $t \geq s(x)$.*

Proof. Killing rigidity implies that, at every $x \in M$, the maps π_{k+t}^{k+t+1} define, for t sufficiently large, a descending sequence of finite dimensional vector spaces. Such sequence of spaces must stabilize at some s that depends on x.

6. Pseudo-Riemannian metrics, complete parallelisms, and connections

A complete parallelism on a manifold is an ordered set of smooth vector fields which forms a basis of the tangent space at every point of the manifold. Such collection of vector fields induces a trivialization of the first order frame bundle and it defines a geometric structure of order 1 and algebraic type on the manifold. A diffeomorphism of the manifold induces an automorphism of such structure if and only if it fixes each vector field (there is an obvious corresponding claim for infinitesimal automorphisms).

A connection on the k-th order frame bundle $L^{(k)}(M)$ of a manifold M defines a geometric structure of order $k+1$ which can be proved to be of algebraic type. For such connection we define the standard horizontal vector field on $L^{(k)}(M)$ associated to $v \in \mathbb{R}^n$ as the unique horizontal field $B(v)$ such that for every $\alpha \in L^{(k)}(M)$ we have $d\pi_\alpha(B(v)_\alpha) = \alpha_1(v)$, where $\pi\colon L^{(k)}(M) \to M$ is the canonical map and $\alpha_1 \in L(M)$ is the projection of α into $L(M)$ (cf. [7]). A diffeomorphism of M induces an automorphism of a connection if and only if its induced map on $L^{(k)}(M)$ fixes each standard horizontal vector field. A similar condition is satisfied for infinitesimal automorphisms.

A pseudo-Riemannian metric or metric tensor defines a geometric structure of order 1 and algebraic type given by a reduction of the first frame bundle to the algebraic subgroup of orthogonal transformations of a suitable scalar product.

PROPOSITION 6.1. *If σ is a complete parallelism on M, then the group $\mathrm{Aut}^r(\sigma, x)$ is trivial for all $r \geq 1$ and all $x \in M$. In particular, σ is 0-rigid.*

Proof. Let $j^1_x(\varphi) \in \mathrm{Aut}^1(\sigma, x)$. Then $d\varphi_x$ fixes a basis of T_xM and hence $j^1_x(\varphi)$ is the identity element. Thus $\mathrm{Aut}^1(\sigma, x)$ is trivial.

Let $X_1, \ldots, X_n$ denote the vector fields that define the parallelism on M. If $j^2_x(\varphi) \in \mathrm{Aut}^2(\sigma, x)$ satisfies $j^1_x(\varphi) = e$, then a straightforward computation (using the fact that $d\varphi(X_i) = X_i \circ \varphi$ up to order 2 at x, for every i) shows that $j^2_x(\varphi) = e$ is trivial. In particular, σ is 0-rigid and so it is r-rigid for every $r \geq 0$. It follows that $\mathrm{Aut}^r(\sigma, x)$ is trivial for every $r \geq 1$ since it is a subgroup of $\mathrm{Aut}^1(\sigma, x)$.

This result is amplified by the following proposition.

PROPOSITION 6.2. *A complete parallelism on $L^{(k)}(M)$ defines a geometric structure σ of order $k+1$ and algebraic type on M. Moreover, for every $x \in M$ and $r \geq 0$, the group $\mathrm{Aut}^{k+r+1}(\sigma, x)$ is trivial. In particular, σ is 0-rigid.*

Proof. The first part of this statement is elementary; it will be shown that the automorphism groups $\mathrm{Aut}^{k+r+1}(\sigma, x)$ are trivial.

Let σ_0 denote the geometric structure on $L^{(k)}(M)$ induced by the complete parallelism. Since an automorphism of both σ and σ_0 is determined by the condition of preserving the parallelism up to a certain order, it is easily seen that, for every $r \geq 0$, an element $j_x^{k+r+1}(\varphi) \in D_x^{(k+r+1)}(M)$ belongs to $\mathrm{Aut}^{k+r+1}(\sigma, x)$ if and only if $j_\alpha^{r+1}(\varphi_{(k)}) \in \mathrm{Aut}^{r+1}(\sigma_0, \alpha)$ for some (and hence every) $\alpha \in \pi^{-1}(x)$, where π denotes the canonical projection $L^{(k)}(M) \to M$. The result now follows from Proposition 6.1.

As for connections and metrics we have the following two results.

PROPOSITION 6.3. *A smooth connection on $L^{(k)}(M)$ is a 0-rigid structure on M.*

Proof. It follows from the fact that an infinitesimal automorphism of a connection preserves, up to a certain order, a complete parallelism on $L^{(k)}(M)$ consisting of horizontal and vertical fields.

PROPOSITION 6.4. *A smooth metric tensor g on M is 0-rigid. In particular, for every $r \geq 1$ and all $x \in M$, the natural projection $\pi_1^r \colon \mathrm{Aut}^r(g, x) \to \mathrm{Aut}^1(g, x)$ is injective.*

Proof. Let $j_x^2(\varphi) \in \mathrm{Aut}^2(g, x)$ be such that $j_x^1(\varphi) = e$. In particular, $\varphi_{(1)}$ fixes the points in the fiber of $L(M)$ above x. Hence, if σ is the geometric structure on $L(M)$ induced by the complete parallelism defined by horizontal vector fields with respect to the Levi-Civita connection of g and vertical fields coming from bases of $\mathbb{R}^n$ and $\mathfrak{gl}(n)$, then $j_\alpha^1(\varphi_{(1)}) \in \mathrm{Aut}^1(\sigma, \alpha)$ for every α in the fiber of $L(M)$ over x. By Proposition 6.1 the group $\mathrm{Aut}^1(\sigma, \alpha)$ is trivial and so $j_\alpha^1(\varphi_{(1)}) = e$, from what it follows that $j_x^2(\varphi) = e$.

A general method for extending certain order 1 infinitesimal Killing fields for a metric tensor was developed by Nomizu in [11]. Some of his results will be presently reviewed and used to prove that infinitesimal Killing fields (of sufficiently large order) for a parallelism on a frame bundle extend to local Killing fields. We observe that the results from Nomizu [11] are stated for Riemannian metrics, but those considered here hold for any metric tensor with the same proof.

Let g be a smooth metric tensor on M with associated Levi-Civita connection ∇. Then the assignment

$$j_x^1(X) \mapsto (X_x, -\nabla X|_x)$$

realizes the space $T_x^{(2)}M$ of 1-jets of vector fields at x as the set of pairs (v, A) where $v \in T_xM$ and A is a linear operator on T_xM. In what follows, $T_x^{(2)}M$ will be identified with the set of such pairs. This identification establishes a correspondence between the set of infinitesimal Killing vector fields of order 1 at x and the set $\mathbb{K}(x)$ of pairs (v, A), where A is anti-symmetric with respect to g_x.

DEFINITION 6.5. A Killing generator at a point $x \in M$ is an infinitesimal Killing vector field $(v, A) \in \mathbb{K}(x)$ of order 1 which has the following property: if A is extended to a derivation $\mathfrak{A}$ of the tensor algebra of T_xM, this extension satisfies

$$\mathfrak{A}(\nabla^m R) = -\nabla_v(\nabla^m R),$$

for every integer $m \geq 0$. The set of Killing generators at x is denoted by $\mathcal{K}(x)$.

Killing generators were introduced by Nomizu [11]. Their relevance to the problem of extending infinitesimal Killing fields is made evident by the following result.

PROPOSITION 6.6. *Let M be an analytic manifold and let g be an analytic metric tensor on M. Then for every $x \in M$ and every Killing generator (v, A), there is a unique analytic local Killing field X defined on a neighborhood of x such that $j^1_x(X) = (v, A)$.*

Proof. The claim is a direct consequence of Theorems 3 and 4 and Lemma 3 in [11].

LEMMA 6.7. *Let g be a smooth metric tensor on M with Levi-Civita connection ∇ and let X be a vector field defined in a neighborhood of a point $x \in M$. Let $\mathfrak{T}(x)$ be the tensor algebra of T_xM and let $\mathfrak{A}\colon \mathfrak{T}(x) \to \mathfrak{T}(x)$ be the map given by:*

$$\mathfrak{A}(K) = L_X\widetilde{K} - \nabla_X\widetilde{K},$$

where $\widetilde{K}$ is a tensor on TM of the same type as K and that extends K to a neighborhood of x. Then $\mathfrak{A}$ is a well defined derivation of $\mathfrak{T}(x)$ that extends the linear map $-\nabla X|_x$.

Proof. Since ∇ is a torsion-free connection, the identity $-\nabla_Y X = L_X Y - \nabla_X Y$ holds for every local field Y defined in a neighborhood of x. Hence, as both L_X and ∇_X are derivations of the tensor algebra of TM, all that needs to be shown is that $\mathfrak{A}$ is well defined. If f is a smooth function and $\widetilde{K}$ is a tensor on TM, both defined in a neighborhood of x, then

$$\begin{aligned}(L_X - \nabla_X)(f\widetilde{K}) &= X(f)\widetilde{K} + fL_X\widetilde{K} - X(f)\widetilde{K} - f\nabla_X\widetilde{K} \\ &= f(L_X - \nabla_X)(\widetilde{K}).\end{aligned}$$

This implies, by a standard argument, that the expression for $\mathfrak{A}(K)$ in the statement does not depend on the choice of $\widetilde{K}$.

The next result shows that infinitesimal Killing fields of sufficiently large order provide Killing generators.

PROPOSITION 6.8. *Let g be a smooth metric tensor on M with Levi-Civita connection ∇. For every $x \in M$, there is an integer $r(x) \geq 1$ such that if $r \geq r(x)$ and $j_x^r(X) \in \mathrm{Kill}^r(g,x)$, then the order 1 infinitesimal Killing field $j_x^1(X) = (X_x, -\nabla X|_x)$ is a Killing generator.*

Proof. Let $x \in M$ and let $\mathcal{K}_m(x)$ be the subspace of $\mathbb{K}(x)$ consisting of pairs (v, A) such that when A is extended to a derivation $\mathfrak{A}$ on the tensor algebra of T_xM we have $\mathfrak{A}(\nabla^l R) = -\nabla_v(\nabla^l R)$, for every integer $l = 0, \ldots, m$. Since $\mathbb{K}(x)$ is finite dimensional, the descending sequence of vectors spaces $\mathcal{K}_m(x)$ must stabilize: there is an integer $m(x)$ such that $\mathcal{K}_m(x) = \mathcal{K}_{m(x)}(x)$ for $m \geq m(x)$. In particular, an element $(v, A) \in \mathbb{K}(x)$ is a Killing generator if and only if the identity $A(\nabla^m R) = -\nabla_v(\nabla^m R)$ holds for every $m = 0, \ldots, m(x)$.

We now show that if $r \geq m(x) + 3$ and $j_x^r(X) \in \mathrm{Kill}^r(g,x)$, then $j_x^1(X) = (X_x, -\nabla X|_x)$ is a Killing generator. Since $j_x^r(X)$ is an infinitesimal Killing field of order greater than or equal to $m(x) + 3$, it is easily seen that $L_X(\nabla^m R) = 0$ at x for every $m = 0, \ldots, m(x)$. Hence, by Lemma 6.7, if $\mathfrak{A}$ denotes the derivation on the tensor algebra of T_xM that extends $-\nabla X|_x$, then $\mathfrak{A}(\nabla^m R) = -\nabla_v(\nabla^m R)$ at x for every $m = 0, \ldots, m(x)$, and thus $j_x^1(X) = (X_x, -\nabla X|_x)$ is a Killing generator.

The previous discussion implies the following result that ensures the extension of infinitesimal Killing fields to local ones for metric tensors.

PROPOSITION 6.9. *Let M be an analytic manifold, and let g be an analytic metric tensor on M with Levi-Civita connection ∇. For every $x \in M$ there is an integer $r(x)$ with the following property: if $r \geq r(x)$ and $j_x^r(X) \in \mathrm{Kill}^r(g,x)$, then there is a unique analytic local Killing vector field Y defined in a neighborhood of x such that $j_x^r(Y) = j_x^r(X)$.*

Proof. Let $r(x)$ be as in Proposition 6.8 and let $j_x^r(X) \in \mathrm{Kill}^r(g,x)$ with $r \geq r(x)$. Then $j_x^1(X) = (X_x, -\nabla X|_x)$ is a Killing generator and by Proposition 6.6 there is an analytic local Killing field Y defined in a neighborhood of x such that $j_x^1(Y) = (X_x, -\nabla X|_x)$. In particular, $j_x^r(Y) \in \mathrm{Kill}^r(g,x)$ and so by Proposition 6.4 the condition $j_x^1(Y) = j_x^1(X)$ implies $j_x^r(Y) = j_x^r(X)$. Uniqueness follows from that in Proposition 6.6.

We also obtain the following corresponding results for complete parallelisms and connections.

PROPOSITION 6.10. *Let M be an analytic manifold. Let σ be an analytic, complete parallelism on the k-th order frame bundle $L^{(k)}(M)$, considered as a geometric structure of order $k+1$ on M. For every*

$x \in M$ there is an integer $r(x)$ such that if $r \geq r(x)$ and $j^r_x(X) \in$ $\mathrm{Kill}^r(\sigma, x)$, then there is a unique analytic local Killing vector field Y defined in a neighborhood of x so that $j^r_x(Y) = j^r_x(X)$.

Proof. Let g denote the unique analytic Riemannian metric tensor on $L^{(k)}(M)$ for which the vector fields that define σ form an orthonormal base at every point. Let $x \in M$, let $\alpha(x)$ be an element in the fiber of $L^{(k)}(M)$ above x, and let $r(\alpha(x))$ be the integer given by Proposition 6.9. Define $r(x) = \max(r(\alpha(x)) + k, s(x))$, where $s(x)$ is given by Corollary 5.6.

Let $r \geq r(x)$ and choose $j^r_x(X) \in \mathrm{Kill}^r(\sigma, x)$. Then the local flow of $X_{(k)}$ preserves the complete parallelism σ up to order $r-k$ at $\alpha(x)$ and it follows that $j^{r-k}_{\alpha(x)}(X_{(k)}) \in \mathrm{Kill}^{r-k}(g, \alpha(x))$. Hence, by the choice of r, Proposition 6.9 ensures the existence of an analytic vector field Z defined in a neighborhood of $\alpha(x)$ in $L^{(k)}(M)$, which is a Killing field (on its domain) for the metric tensor g and which satisfies $j^{r-k}_{\alpha(x)}(X_{(k)}) = j^{r-k}_{\alpha(x)}(Z)$.

By Corollary 5.6 and the choice of r, for every $l \geq r$ there is a unique infinitesimal Killing field $j^l_x(X^l) \in \mathrm{Kill}^l(\sigma, x)$ such that $j^r_x(X^l) = j^r_x(X)$. As before $j^{l-k}_{\alpha(x)}(X^l_{(k)}) \in \mathrm{Kill}^{l-k}(g, \alpha(x))$, and since $j^{l-k}_{\alpha(x)}(Z) \in \mathrm{Kill}^{l-k}(g, \alpha(x))$ as well, both with the same $(r-k)$-jet, the rigidity of g implies (for l large enough) that $j^{l-k}_{\alpha(x)}(X^l_{(k)}) = j^{l-k}_{\alpha(x)}(Z)$. It follows from Lemma 4.4 that Z satisfies the conditions stated in such lemma up to order $(l-k)$ at $\alpha(x)$ for every l sufficiently large. Since Z, the canonical form of $L^{(k)}(M)$ and the $\mathrm{Gl}^{(k)}(n)$-action are all analytic it follows that Z satisfies the conditions of Lemma 4.4 in its domain, and so there is a vector field Y defined in a neighborhood of x in M such that $Y_{(k)} = Z$. This last identity shows that $Y_{(k)}$ preserves the parallelism on $L^{(k)}(M)$ and so Y is a Killing field for σ in a neighborhood of x. The identities $j^{r-k}_{\alpha(x)}(Y) = j^{r-k}_{\alpha(x)}(Z) = j^{r-k}_{\alpha(x)}(X_{(k)})$ imply $j^r_x(Y) = j^r_x(X)$, and so Y is the required extension.

Uniqueness is proved from the corresponding one for g by a similar argument using analyticity.

PROPOSITION 6.11. *Let M be an analytic manifold carrying an analytic connection ω on its k-th order frame bundle $L^{(k)}(M)$. For every $x \in M$ there is an integer $r(x)$ such that if $r \geq r(x)$ and $j^r_x(X) \in$ $\mathrm{Kill}^r(\omega, x)$, then there is a unique analytic local Killing vector field Y defined in a neighborhood of x so that $j^r_x(Y) = j^r_x(X)$.*

Proof. This is proven with similar arguments, using the parallelism defined by a connection with horizontal and vertical vector fields.

7. Structures of finite type

This section shows that, in the analytic category, infinitesimal Killing fields of geometric structures of finite type have unique local extensions. Those properties of geometric structures of finite type that we state without proof can be found in Kobayashi [8].

DEFINITION 7.1. Let $\mathfrak{h}$ be a Lie subalgebra of the general linear algebra $\mathfrak{gl}(n)$ and let k be a non-negative integer. The k-th prolongation of $\mathfrak{h}$ is the vector space $\mathfrak{h}_k$ of $\mathbb{R}^n$-valued symmetric $(k+1)$-multilinear maps T on $\mathbb{R}^n$ so that for every $v_1, \dots, v_k \in \mathbb{R}^n$ the linear map $v \mapsto T(v_1, \dots, v_k, v)$ belongs to $\mathfrak{h}$. We will say that $\mathfrak{h}$ is of finite type if $\mathfrak{h}_k = 0$ for some integer k.

Notice that $\mathfrak{h}_0 = \mathfrak{h}$ for any Lie algebra. Also we will denote $\mathfrak{h}_{-1} = \mathbb{R}^n$. For $k \geq 1$, we will consider $\mathfrak{h}_k$ as a Lie algebra with the trivial abelian structure.

DEFINITION 7.2. An H-structure of order 1 on a smooth manifold M is called of finite type if the Lie algebra of H is of finite type. A geometric structure of order 1 defined by a reduction to a closed subgroup will be called of finite type if the reduction is of finite type.

We recall that a metric tensor defines a geometric structure of finite type. We refer to Kobayashi [8] for this and other examples.

Let P be an H-reduction of $L(M)$ and denote with θ the canonical form of P. A subspace $\mathcal{H}$ of $T_\alpha P$, for some $\alpha \in P$, is called a horizontal space tangent to P at α if $\theta|_{\mathcal{H}}: \mathcal{H} \to \mathbb{R}^n$ is an isomorphism. For every such horizontal space we denote with $c(\alpha, \mathcal{H})$ the antisymmetric bilinear map on $\mathbb{R}^n$ given by:

$$c(\alpha, \mathcal{H})(v_1, v_2) = d\theta_\alpha(\theta|_{\mathcal{H}}^{-1}(v_1), \theta|_{\mathcal{H}}^{-1}(v_2))$$

for every $v_1, v_2 \in \mathbb{R}^n$.

Consider the linear map $\partial: \mathfrak{h} \otimes \mathbb{R}^{n*} \to \mathbb{R}^n \otimes \wedge^2 \mathbb{R}^{n*}$ defined so that for every $f \in \mathfrak{h} \otimes \mathbb{R}^{n*}$ we have:

$$\partial f(v_1, v_2) = \frac{1}{2}(f(v_1)v_2 - f(v_2)v_1)$$

where $v_1, v_2 \in \mathbb{R}^n$. For the rest of this section we will fix a vector space $\mathcal{C}$ such that $\mathbb{R}^n \otimes \wedge^2 \mathbb{R}^{n*} = \partial(\mathfrak{h} \otimes \mathbb{R}^{n*}) \oplus \mathcal{C}$.

Based on this we introduce the concept of prolongation of a reduction as found in Kobayashi [8].

DEFINITION 7.3. Let P be a smooth H-reduction of $L(M)$. The first prolongation P_1 of P is the subset of $L(P)$ consisting of frames $\mathbb{R}^n \oplus \mathfrak{h} \to T_\alpha P$ of the form $(v, X) \mapsto \theta|_{\mathcal{H}}^{-1}(v) + X_\alpha^*$, where $\mathcal{H}$ is a horizontal space tangent to P at α such that $c(\alpha, \mathcal{H}) \in \mathcal{C}$.

From now on, if H is a subgroup of $\mathrm{Gl}(n)$ with Lie algebra $\mathfrak{h}$, then for every $k \geq 1$ we will denote with H_k the subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h}_0 \oplus \ldots \oplus \mathfrak{h}_{k-1})$ consisting of the transformations of the form $(v, X) \mapsto (v, T(v) + X)$, where $X \in \mathfrak{h}_0 \oplus \ldots \oplus \mathfrak{h}_{k-1}$ and $T \in \mathfrak{h}_k$ is considered as a linear map $\mathbb{R}^n \to \mathfrak{h}_{k-1}$. The group H_k is called the k–th prolongation of H. It is easy to check that H_k is a simply connected, closed, abelian Lie subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h}_0 \oplus \ldots \oplus \mathfrak{h}_{k-1})$ with Lie algebra $\mathfrak{h}_k$.

The following result is a direct consequence of the discussion found in Kobayashi [8].

PROPOSITION 7.4. *Let P be a smooth H-reduction of the first frame bundle $L(M)$ of a manifold M. Then the first prolongation P_1 is a smooth reduction of $L(P)$ to the subgroup H_1.*

Given this result, for any principal bundle P as above we define by induction its k-th order prolongation P_k as the reduction of $L(P_{k-1})$ provided by the first prolongation of P_{k-1}. For such setup, we will also denote $P_0 = P$ and $P_{-1} = M$. From Kobayashi [8] it is known that, for every $k \geq 0$, the bundle P_{k+1} is a reduction of $L(P_k)$ to the closed subgroup H_{k+1} and so it defines a geometric structure on P_k which we will denote with σ_{k+1}. If H is closed and σ is the geometric structure defined by P, then we will also denote $\sigma_0 = \sigma$ to be consistent with our notation. Since H_k is a vector group isomorphic to $\mathfrak{h}_k$, if P is an H-structure of finite type, then there is a smallest integer r for which P_{r+1} defines a trivialization of $L(P_r)$, i.e. a complete parallelism on P_r.

By considering the prolongations of a reduction, the following result gives a useful construction from which most of the basic local properties of geometric structures of finite type can be obtained. The proof of this proposition is a direct consequence of the properties of prolongations of reductions discussed in Kobayashi [8].

PROPOSITION 7.5. *Let P be an H-reduction of the principal frame bundle $L(M)$ of M. If H is closed and σ is the geometric structure defined by P, then for every Killing vector field X defined on an open subset of M there is a vector field X_i on P_i for every $i \geq -1$, with $X_{-1} = X$, such that:*

1. *X_i is the restriction to P_i of the vector field $(X_{i-1})_{(1)}$ on $L(P_{i-1})$, for $i \geq 0$,*

2. *X_i is a Killing field for the geometric structure σ_{i+1} on P_i defined by P_{i+1}, for $i \geq -1$.*

In particular, if σ is of finite type and r is the first integer for which the prolongation P_{r+1} is a trivialization of $L(P_r)$, then X_r is a Killing field for the corresponding complete parallelism of P_r.

We will need a similar result for infinitesimal Killing fields of geometric structures of finite type. We achieve this with an alternative description of the first prolongation of a reduction. This in turn will require the notion of principal prolongation, as found in [9]. Here we state some of its properties and refer to [9] for the corresponding proofs.

DEFINITION 7.6. Let P be a smooth H-reduction of the frame bundle $L(M)$ of a manifold M. The principal prolongation of P is the set $W^1(P)$ of 1-jets at $(0, I)$ of local principal bundle diffeomorphisms from $\mathbb{R}^n \times H$ onto open subbundles of P.

In what follows, we consider $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$ as a Lie group with the semidirect product structure given by $(A, T)(A', T') = (AA', T \circ A' + T')$, where both T, T' are seen as linear maps $\mathbb{R}^n \to \mathfrak{h}$. Observe that the injective homomorphism $\rho\colon \mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*} \to \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$ given by $\rho(A, T)(v, X) = (Av, Tv + X)$ realizes $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$ as a closed subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$ containing H_1. The group $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$ can equivalently be defined as the space of 1-jets at $(0, I)$ of local principal bundle diffeomorphisms of $\mathbb{R}^n \times H$ which fix $(0, I)$. Such description provides the natural right action considered in the following result, whose proof is a direct consequence of the theory from [9].

PROPOSITION 7.7. *Let P be a smooth H-reduction of the frame bundle $L(M)$ of a manifold M. Then $W^1(P)$ is a smooth principal bundle over P with structure group $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$. Moreover, the map:*

$$\begin{aligned} W^1(P) &\to \; L(P) \\ j^1_{(0,I)}(\lambda) &\mapsto \; d\lambda_{(0,I)} \end{aligned}$$

where $d\lambda_{(0,I)}\colon \mathbb{R}^n \oplus \mathfrak{h} \to T_{\lambda(0,I)}P$ is considered as a frame of order 1 of P, realizes $W^1(P)$ as a smooth $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$-reduction of $L(P)$.

Observe that the map:

$$\begin{aligned} (\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*})/H_1 &\to \; \mathrm{Gl}(n) \times (\mathbb{R}^n \otimes \wedge^2 \mathbb{R}^{n*})/\mathcal{C} \\ (A, T)H_1 &\mapsto \; (A, \partial(T) + \mathcal{C}) \end{aligned}$$

is a $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$-equivariant diffeomorphism if we consider the smooth action of $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$ on $\mathrm{Gl}(n) \times (\mathbb{R}^n \otimes \wedge^2 \mathbb{R}^{n*})/\mathcal{C}$ given by $(A,T)(B,\Lambda+\mathcal{C}) = (AB, \Lambda + \partial(T\circ B)+\mathcal{C})$. With respect to such diffeomorphism, the next result shows that the first prolongation of a reduction can be obtained as a further reduction of the principal prolongation. We also prove that the map σ' below determines the automorphisms of σ_1 among the bundle maps.

LEMMA 7.8. *Let P be a smooth H-reduction of the frame bundle $L(M)$ of a manifold M and $\mathcal{C}$ be a vector space as in Definition 7.3. Let σ' be the smooth map given by:*

$$\begin{aligned}\sigma' \colon W^1(P) \to &\quad \mathrm{Gl}(n)\times(\mathbb{R}^n\otimes\wedge^2\mathbb{R}^{n*})/\mathcal{C}\\ j^1_{(0,I)}(\lambda) \mapsto &\ (d\bar{\lambda}_0^{-1}\circ\lambda(0,I), c(\lambda(0,I),\mathcal{H}_\lambda)+\mathcal{C})\end{aligned}$$

where $\bar{\lambda}$ is the local diffeomorphism covered by λ and $\mathcal{H}_\lambda$ is the horizontal space tangent to P given by the image of the differential at 0 of the map $x\mapsto\hat{\lambda}(x)=\lambda(x,I)$. Then the map σ' is $\mathrm{Gl}(n)\ltimes\mathfrak{h}\otimes\mathbb{R}^{n*}$-equivariant and it realizes P_1 as the H_1-reduction of $W^1(P)$ given by $P_1 = \sigma'^{-1}(I,\mathcal{C})$.

Proof. Let $j^1_{(0,I)}(\lambda)\in W^1(P)$. Then $j^1_{(0,I)}(\lambda)$ defines the element of $L(P)$ given by $(v,X)\mapsto d\hat{\lambda}_0(v)+X^*_{\lambda(0,I)}$. On the other hand, a straightforward computation shows that $\theta|^{-1}_{\mathcal{H}_\lambda} = d\hat{\lambda}_0\circ d\bar{\lambda}_0^{-1}\circ\lambda(0,I)$. From these expressions and the definition of $c(\lambda(0,I),\mathcal{H}_\lambda)$ we conclude that $j^1_{(0,I)}(\lambda)\in P_1$ if and only if $\sigma'(j^1_{(0,I)}(\lambda)) = (I,\mathcal{C})$.

On the other hand, to prove that σ' is $\mathrm{Gl}(n)\ltimes\mathfrak{h}\otimes\mathbb{R}^{n*}$-equivariant, we choose $j^1_{(0,I)}(\xi)\in W^1(P)$ lying in the same fiber of $j^1_{(0,I)}(\lambda)$. Then $j^1_{(0,I)}(\xi) = j^1_{(0,I)}(\lambda)(A,T)$ for some $(A,T)\in\mathrm{Gl}(n)\ltimes\mathfrak{h}\otimes\mathbb{R}^{n*}$ and it is easy to check that this implies:

$$\begin{aligned}d\hat{\xi}_0 &= d\hat{\lambda}_0 A + *_\alpha T\\ d\bar{\xi}_0 &= d\bar{\lambda}_0 A\end{aligned}$$

where $\alpha=\lambda(0,I)=\xi(0,I)$ and $*_\alpha$ denotes the map $\mathfrak{h}\to T_\alpha P$ given by $X\mapsto X^*_\alpha$. Now choose a connection ω on P for which $\mathcal{H}_\lambda$ is horizontal. Then the structural equations for ω imply that:

$$d\theta_\alpha(u,v) = -\frac{1}{2}(\omega_\alpha(u)(\theta_\alpha(v)) - \omega_\alpha(v)(\theta_\alpha(u))) + \Theta_\alpha(u,v)$$

for every $u,v\in T_\alpha P$, where Θ denotes the torsion of ω. If we compute $c(\alpha,\mathcal{H}_\xi) = d\theta_\alpha(\theta|^{-1}_{\mathcal{H}_\xi},\theta|^{-1}_{\mathcal{H}_\xi})$ with the above structural equation, replace the formula $\theta|^{-1}_{\mathcal{H}_\xi} = d\hat{\xi}_0\circ d\bar{\xi}_0^{-1}\circ\alpha$ and use the above expressions for $d\hat{\xi}_0$ and $d\bar{\xi}_0$ and the fact that $\Theta_\alpha = d\theta_\alpha|_{\mathcal{H}_\lambda\times\mathcal{H}_\lambda}$, then we obtain:

$$c(\alpha,\mathcal{H}_\xi) = -\partial(T\circ A^{-1}\circ d\bar{\lambda}_0^{-1}\circ\alpha) + c(\alpha,\mathcal{H}_\lambda)$$

which is easily seen to provide the required equivariance.

LEMMA 7.9. *Let P be a smooth H-reduction of the frame bundle $L(M)$ of a manifold M and let σ' be as in Lemma 7.8. If Φ is a bundle diffeomorphism of P defined in a neighborhood of $\alpha_0 \in P$ and $\alpha_1 \in W^1(P)$ lies in the fiber of α_0, then $\sigma' \circ \Phi_{(1)} = \sigma'$ up to order $s-1$ at α_1 if and only if $j^s_{\alpha_0}(\Phi) \in \mathrm{Kill}^s(\sigma_1, \alpha_0)$.*

Proof. First observe that since Φ is a bundle diffeomorphism of P, its lift $\Phi_{(1)}$ to a neighborhood of α_1 in $L(P)$ leaves invariant $W^1(P)$.

On the other hand, if we consider σ_1 as a $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$-equivariant map $L(P) \to \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})/H_1$ and since $P_1 = \sigma^{-1}(eH_1)$, then we have $\sigma_1(\alpha g) = g^{-1}H_1$ for every $\alpha \in P_1$ and $g \in \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$. The corresponding property for σ' together with the identification $(\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*})/H_1 \cong \mathrm{Gl}(n) \times (\mathbb{R}^n \otimes \wedge^2\mathbb{R}^{n*})/\mathcal{C}$ shows that $\sigma_1|_{W^1(P)} = \sigma'$. This proves that, if $j^s_{\alpha_0}(\Phi) \in \mathrm{Kill}^s(\sigma_1, \alpha_0)$, then $\sigma' \circ \Phi_{(1)} = \sigma'$ up to order $s-1$ at α_1.

For the converse, observe that:

$$\sigma_1 \circ \Phi_{(1)}(\alpha g) = g^{-1}\sigma_1 \circ \Phi_{(1)}(\alpha) = g^{-1}\sigma' \circ \Phi_{(1)}(\alpha)$$

for every $\alpha \in W^1(P)$ and $g \in \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$. Using this, the identity $\sigma_1(\alpha g) = g^{-1}\sigma_1(\alpha) = g^{-1}\sigma'(\alpha)$, for $\alpha \in W^1(P)$ and $g \in \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$, and the fact that the map $W^1(P) \times \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h}) \to L(P)$ given by $(\alpha, g) \mapsto \alpha g$ is a surjective submersion, a straightforward computation shows that $\sigma' \circ \Phi_{(1)} = \sigma'$ up to order $s-1$ at α_1 implies $\sigma_1 \circ \Phi_{(1)} = \sigma_1$ up to order $s-1$ at α_1 and so $j^s_{\alpha_0}(\Phi) \in \mathrm{Kill}^s(\sigma_1, \alpha_0)$.

We will need the following result that provides a way to suitably lift vector fields to reductions when dealing with infinitesimal Killing fields.

LEMMA 7.10. *For H closed, let P be a smooth H-reduction of $L(M)$ that defines a geometric structure σ on a manifold M, let $j^s_{x_0}(X) \in \mathrm{Kill}^s(\sigma, x_0)$ and let φ_t be the local flow of X in a neighborhood of $x_0 \in M$. Then for every α_0 in the fiber of P over x_0, there is a smooth map $\Phi\colon J \times U \to L(M)$ defined in a neighborhood of $(0, \alpha_0)$ in $\mathbb{R} \times L(M)$, with $U = L(\bar{U})$ for some open set $\bar{U} \subset M$, such that for every $t \in J$:*

1. *Φ_t is a local principal bundle diffeomorphism of $L(M)$ covering φ_t,*
2. *$\Phi_t(U \cap P) \subset P$,*
3. *$\Phi_t = (\varphi_t)_{(1)}$ up to order $s-1$ at α_0,*

where we denote $\Phi_t(\alpha) = \Phi(t, \alpha)$, for every $(t, \alpha) \in J \times U$.

Proof. We consider σ as a map $L(M) \to \mathrm{Gl}(n)/H$ and so we have $P = \sigma^{-1}(eH)$.

By locally trivializing P in a neighborhood of x_0, we can find $\bar{V}$ and $V = L(\bar{V})$ open neighborhoods of x_0 and α_0 in M and $L(M)$, respectively, together with diffeomorphisms $\bar{\lambda}\colon \mathbb{R}^n \to \bar{V}$ and $\lambda\colon \mathbb{R}^n \times \mathrm{Gl}(n) \to V$ such that λ is a principal bundle diffeomorphism covering $\bar{\lambda}$ and also such that λ satisfies $\lambda(\mathbb{R}^n \times H) = P \cap V$ and $\lambda(0, I) = \alpha_0$. We also choose an open neighborhood $\bar{U}$ of x_0 and an open interval $J \subset \mathbb{R}$ containing 0 such that $\varphi_t(\bar{U}) \subset \bar{V}$, for every $t \in J$. Also, in what follows we consider φ_t restricted to $\bar{U}$, for every $t \in J$.

Consider the map $\widetilde{\sigma} = \sigma \circ \lambda\colon \mathbb{R}^n \times \mathrm{Gl}(n) \to \mathrm{Gl}(n)/H$. Then a direct computation shows that $\widetilde{\sigma}$ is given by $\widetilde{\sigma}(x, A) = A^{-1}H$, for every $(x, A) \in \mathbb{R}^n \times \mathrm{Gl}(n)$.

For very $t \in J$, consider the principal bundle diffeomorphism $\eta_t = \lambda^{-1} \circ (\varphi_t)_{(1)} \circ \lambda$ which covers the diffeomorphism $\bar{\eta}_t = \bar{\lambda}^{-1} \circ \varphi_t \circ \bar{\lambda}$. Hence, for every $t \in J$, there is a smooth map $\hat{\eta}_t$ from a neighborhood of 0 in $\mathbb{R}^n$ into $\mathrm{Gl}(n)$ such that $\eta_t(x, A) = (\bar{\eta}_t(x), \hat{\eta}_t(x)A)$, for every x in neighborhood of 0 in $\mathbb{R}^n$ and every $A \in \mathrm{Gl}(n)$. Note that the maps $(t, x) \mapsto \bar{\eta}_t(x)$ and $(t, x) \mapsto \hat{\eta}_t(x)$ are both smooth in a neighborhood of $(0, 0)$ in $J \times \mathbb{R}^n$.

Fix $t \in J$. Since $j^s_{x_0}(\varphi_t)$ is an infinitesimal automorphism, we know that $\sigma \circ (\varphi_t)_{(1)} = \sigma$ up to order $s - 1$ at α_0. From this it follows that $\widetilde{\sigma} \circ \eta_t = \widetilde{\sigma}$ up to order $s - 1$ at $(0, I)$. Hence the above expression for $\widetilde{\sigma}$ shows that the map $x \mapsto \hat{\eta}_t(x)H$ equals the constant map $x \mapsto eH$ up to order $s - 1$ at 0. We can use a trivialization of the submersion $\mathrm{Gl}(n) \to \mathrm{Gl}(n)/H$ and local coordinates for $\mathrm{Gl}(n)$ so that H corresponds to $\mathbb{R}^k \times \{0\}$ (the first k coordinates, $k = \dim H$) to conclude that, with respect to such coordinates, the partial derivatives from order 1 up to order $s - 1$ of the last $n^2 - k$ components of $\hat{\eta}_t$ vanish at 0. Then a suitable Taylor polynomial of $\hat{\eta}_t$ up to order $s - 1$ at 0 can be used to define a smooth map $\hat{\xi}_t$ from a neighborhood of 0 in $\mathbb{R}^n$ into H that equals $\hat{\eta}_t$ up to order $s - 1$ at 0. Note that our construction is such that the map $(t, x) \mapsto \hat{\xi}_t(x)$ is smooth.

From the above it follows that, for every $t \in J$, the smooth map $\xi_t(x, A) = (\bar{\eta}_t(x), \hat{\xi}_t(x)A)$ defined in a neighborhood of $(0, I)$ in $\mathbb{R}^n \times \mathrm{Gl}(n)$ is a principal bundle diffeomorphism that maps $\mathbb{R}^n \times H$ onto itself and that equals η_t up to order $s - 1$ at $(0, I)$. Also note that ξ_t covers $\bar{\eta}_t$. Then the maps $\Phi_t = \lambda \circ \xi_t \circ \lambda^{-1}$ provide the required principal bundle diffeomorphisms on $U = L(\bar{U})$.

The next result for infinitesimal Killing fields is similar to Proposition 7.5.

PROPOSITION 7.11. *For H closed, let P be a smooth H-reduction of the principal frame bundle $L(M)$ of M and σ the corresponding geometric structure. Let $x_{-1} = x \in M$ and for every $i \geq 0$ choose $x_i \in P_i$ lying in the fiber of P_i over x_{i-1}. Then for every infinitesimal Killing field $j^s_x(X) \in \mathrm{Kill}^s(\sigma, x)$, there is a vector field X_i defined in a neighborhood of x_i in P_i for every $i = -1, \dots, s$, with $X_{-1} = X$, such that:*

1. *$j^{s-i+1}_{x_i}(X_i)$ is an infinitesimal Killing field for the geometric structure σ_{i+1} on P_i defined by P_{i+1}, for $i = -1, \dots, s$,*
2. *X_i and $(X_{i-1})_{(1)}|_{P_i}$ coincide up to order $s - i + 1$ at x_i as sections of $TL(P_{i-1})|_{P_i}$, for $i = 0, \dots, s$.*

Moreover, for every $i = 0, \dots, s$, the assignment $j^s_x(X) \mapsto j^{s-i+1}_{x_i}(X_i)$ provides a well defined linear map $\mathrm{Kill}^s(\sigma, x) \to \mathrm{Kill}^{s-i+1}(\sigma_{i+1}, x_i)$. In particular, if σ is of finite type and r is the first integer for which the prolongation P_{r+1} is a trivialization of $L(P_r)$, then for $s \geq r$ and $j^s_x(X) \in \mathrm{Kill}^s(\sigma, x)$ we have that $j^{s-r+1}_{x_r}(X_r)$ is an infinitesimal Killing field for the corresponding complete parallelism of P_r.

Proof. Let $j^s_x(X) \in \mathrm{Kill}^s(\sigma, x)$, denote with φ_t the local flow of X, and consider the smooth maps Φ_t given by Lemma 7.10. In a neighborhood of x_0 in P_0 define the vector field:

$$X_0(\alpha) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t(\alpha)$$

Then by Lemma 7.10, condition 2 above is satisfied for $i = 0$.

Let σ' be the map considered in Lemma 7.8. Using the expression of the bilinear maps $c(\alpha, \mathcal{H})$ in terms of the canonical form of P, that $(\varphi_t)_{(1)}$ preserves such canonical form on $L(M)$ and that Φ_t coincides with $(\varphi_t)_{(1)}$ up to order $s - 1$ at x_0, it is straightforward to check that $(\Phi_t)_{(1)}$ preserves σ' up to order $s - 2$ at x_1. It follows from Lemma 7.9 that Φ_t defines an infinitesimal automorphism of order $s - 1$ of the geometric structure σ_1 on P_0 given by P_1. From this it follows easily that condition 1 is satisfied for $i = 0$.

From the above we can proceed inductively to obtain the vector fields satisfying conditions 1 and 2.

On the other hand, condition 2 can be used inductively to prove that the maps $j^s_x(X) \mapsto j^{s-i+1}_{x_i}(X_i)$ are well defined and linear.

The previous result can be used to prove that the rigidity of complete parallelisms implies Killing rigidity (and thus rigidity) for geometric structures of finite type. Moreover, for H-structures of order 1, with H closed, rigidity and being of finite type are equivalent, as the following result confirms.

PROPOSITION 7.12. *Let σ be a smooth geometric structure of order* 1 *on M defined by a reduction of $L(M)$ to a closed subgroup H of* $\mathrm{Gl}(n)$*. Then σ is of finite type if and only if it is rigid.*

Proof. The direct statement follows from Proposition 7.11 and the rigidity of parallelisms. We will prove the converse statement: that rigidity of the H-structure implies that it is of finite type. For this, by Proposition 5.2, it suffices to show that if $\mathfrak{h}_r \neq 0$, then the action of N^{r+1} on the image of σ^r in $J^r_n(\mathrm{Gl}(n)/H)$ is not free.

Assume thus that $\mathfrak{h}_r \neq 0$, let $L \in \mathfrak{h}_r \setminus 0$ and let $g = (I, 0, \ldots, 0, L) \in N^{r+1} \setminus \{I\}$. Let $j^r(f) \in J^r_n(\mathrm{Gl}(n)/H)$ be a point in the image of σ^r. By means of a local trivialization of $\pi\colon \mathrm{Gl}(n) \to \mathrm{Gl}(n)/H$, we can obtain a map $\widetilde{f}\colon \mathbb{R}^n \to \mathrm{Gl}(n)$ such that $j^r(f) = j^r(\pi(\widetilde{f}))$. Moreover, since σ^r is $\mathrm{Gl}^{(r+1)}(n)$-equivariant, we can choose f so that $f(0) = [I]$ (the class of the identity $I \in \mathrm{Gl}(n)/H$) and thus we can assume that $\widetilde{f}(0) = I$.

We have, by Lemma 2.2, that $gj^r(f) = j^r(\psi f)$. This ψ can be chosen to be defined by $\psi(x)(v) = v + (1/r!)L(v, x, \ldots, x)$, so that $\psi(x) \in \mathrm{Gl}(n) = N^1$ for x in a neighborhood of the origin $0 \in \mathbb{R}^n$. Let h be the $\mathrm{Gl}(n)$-valued map defined in a neighborhood of $0 \in \mathbb{R}^n$ by $h(x) = \widetilde{f}(x)^{-1}\psi(x)\widetilde{f}(x)$. A computation in local coordinates shows that $j^{r-1}(h)$ is the identity element of $J^r_n(\mathrm{Gl}(n))$.

Since $\widetilde{f}(0) = I$ and $L \in \mathfrak{h}_r$, the partial derivatives of order r of h define an $\mathfrak{h}$-valued symmetric r-multilinear transformation on $\mathbb{R}^n$, from what it follows that $j^r(h) \in J^r_n(H)$. In particular, if we consider the action of our given $g \in N^{r+1}$ on $J^r_n(\mathrm{Gl}(n))$ with respect to the homomorphism defined in Lemma 2.2, then $gj^r(\widetilde{f}) = j^r(\psi)j^r(\widetilde{f}) = j^r(\psi\widetilde{f}) = j^r(\widetilde{f}h)$ in $J^r_n(\mathrm{Gl}(n))$. It follows from this that $gj^r(f) = j^r(f)$ in $J^r_n(\mathrm{Gl}(n)/H)$, and thus that the action of N^{r+1} on the image of σ^r is not free, because g is not the identity of N^{r+1}.

From the above properties we now obtain the following result that allows us to extend infinitesimal Killing fields to local Killing fields.

PROPOSITION 7.13. *Let M be an analytic manifold endowed with an analytic geometric structure σ of order* 1 *and finite type defined by a reduction of $L(M)$ to a closed group (e.g., an analytic rigid H-structure of order* 1 *and algebraic type). For every $x \in M$ there is an integer $r(x)$ such that if $r \geq r(x)$ and $j^r_x(X) \in \mathrm{Kill}^r(\sigma, x)$, then there is a unique analytic local Killing vector field Y defined in a neighborhood of x so that $j^r_x(Y) = j^r_x(X)$.*

Proof. This follows from Proposition 7.11 and the arguments used in the proof of Proposition 6.10. The proof uses the fact that the analitycity of σ ensures the analiticity of the prolongations of the associated reduction. We also observe that condition 2 for the vector fields X_i from

Proposition 7.11 is used to ensure that the conditions from Lemma 4.5 are satisfied for X_i up to a suitable finite order at x_i on P_i. The latter allows to use Lemma 4.5 in a way similar to how we used Lemma 4.4 in Proposition 6.10.

8. Algebraic hull of an action

This section reviews the concept of algebraic hull; we refer to Zimmer [15] for more details.

PROPOSITION 8.1. *Let M be a smooth manifold carrying a finite measure μ and let P be a smooth principal H-bundle over M, where H is a real algebraic group. Let G be a Lie group which acts on M ergodically with respect to μ and on P by bundle automorphisms. Then there exist a real algebraic subgroup L of H and a measurable L-reduction Q of P which satisfy the following properties:*

1. *If Q' is a measurable G-invariant L'-reduction of Q such that L' is a real algebraic subgroup of L, then $L' = L$ and $Q' = Q$ almost everywhere (with respect to μ).*
2. *If Q' is a measurable G-invariant L'-reduction of P such that L' is a real algebraic subgroup of H, then there exist $h \in H$ such that $L \subset hL'h^{-1}$ and $Q \subset Q'h^{-1}$ almost everywhere.*

Moreover, if Q_1 and Q_2 are two such reductions with structure groups L_1 and L_2 (respectively), there exist $h \in H$ such that $L_1 = hL_2h^{-1}$ and $Q_1 = Q_2h^{-1}$ almost everywhere.

The conjugacy class of L is called the algebraic hull of the G-action. However, it will often be referred to a real algebraic group L when referring to the algebraic hull.

The following result can be easily obtained from the definition.

PROPOSITION 8.2. *Let P_1 and P_2 be principal fiber bundles over M with real algebraic structure groups H_1 and H_2, respectively, and let $\rho\colon H_1 \to H_2$ be a surjection of real algebraic groups. Assume that a group G acts on M ergodically with respect to a finite measure, and on both P_1 and P_2 by bundle automorphisms. If $f\colon P_1 \to P_2$ is a ρ-equivariant continuous bundle homomorphism that covers the identity and commutes with the action of G, then there exists a measurable L_i-reduction Q_i of P_i where L_i is the algebraic hull of P_i, $i = 1, 2$, such that $f(Q_1) \subset Q_2$ and $\rho(L_1) \subset L_2$ is a Zariski dense subgroup of finite index.*

The computation of the algebraic hull given in the next result is fundamental in the study of actions of simple Lie groups. The details of the proof can be found in Zimmer [16].

PROPOSITION 8.3. *Let G be a connected noncompact simple Lie group acting ergodically on M preserving a finite measure. Let P be the trivial fiber bundle $M \times \mathrm{Gl}(\mathfrak{g})$ where G acts by $g(x, A) = (gx, \mathrm{Ad}_G(g)A)$. Then the algebraic hull of this action is the Zariski closure of $\mathrm{Ad}_G(G)$ in $\mathrm{Gl}(\mathfrak{g})$.*

A geometric structure is not always given by a reduction and so an algebraic hull cannot always be computed. The following results give conditions under which a geometric structure is a reduction.

PROPOSITION 8.4. *Let $\sigma: L^{(k)}(M) \to Q$ be a geometric structure of algebraic type on M. Suppose that a group G acts ergodically on M with respect to a finite measure and preserving σ. Then there is a G-invariant, measurable, conull subset S of M such that $\sigma(L^{(k)}(M)|_S)$ consists of a single $\mathrm{Gl}^{(k)}(n)$-orbit and so it defines a measurable H-reduction for some real algebraic subgroup H of $\mathrm{Gl}^{(k)}(n)$.*

Proof. Note that σ is a G-invariant, $\mathrm{Gl}^{(k)}(n)$-equivariant map, and so it defines a measurable G-invariant map $\widetilde{\sigma}: M \to \mathrm{Gl}^{(k)}(n)\backslash Q$. This map $\widetilde{\sigma}$ must then be essentially constant because $\mathrm{Gl}^{(k)}(n)\backslash Q$ is tame and the action of G on M is ergodic. Thus there exists a G-invariant, conull subset S of M such that $\sigma(L^{(k)}(M)|_S)$ lies in a single orbit of $\mathrm{Gl}^{(k)}(n)$. The rest of the statement follows easily from this.

PROPOSITION 8.5. *Let $\sigma: L^{(k)}(M) \to Q$ be a smooth geometric structure of algebraic type on M. Let G act on M preserving both σ and a finite measure with respect to which almost every orbit is dense (e.g., the measure is smooth and ergodic). Then there is a conull, G-invariant, dense, open subset $U \subset M$ such that the image of σ over U lies in a single $\mathrm{Gl}^{(k)}(n)$-orbit and so it defines a smooth H-reduction for some real algebraic subgroup H of $\mathrm{Gl}^{(k)}(n)$.*

Proof. If $\alpha \in L^{(k)}(M)$ belongs to the fiber over a point in a dense orbit of the action of G in M, then the orbit of α in $L^{(k)}(M)$ under the action of $G \times \mathrm{Gl}^{(k)}(n)$ is dense. Hence the image of σ is contained in the closure of $\mathrm{Gl}^{(k)}(n)\sigma(\alpha)$ in Q. Since the action of $\mathrm{Gl}^{(k)}(n)$ is algebraic, the orbit $\mathrm{Gl}^{(k)}(n)\sigma(\alpha)$ is open in its closure. From this we obtain a G-invariant, open, dense subset U of M such that $\sigma(L^{(k)}(M)|_U)$ is contained in $\mathrm{Gl}^{(k)}(n)\sigma(\alpha)$. This set U must contain every orbit which is dense in M, because it is open and G-invariant. Hence U is also conull because almost every orbit is dense. The conclusion follows from this.

9. Gromov's centralizer theorem

The following result is fundamental in the proof of Gromov's centralizer theorem. It proves the existence of a large number of infinitesimal Killing fields centralizing the action of a suitable group.

LEMMA 9.1. *Let M be a connected analytic manifold on which a connected, noncompact simple Lie group G acts preserving a finite Zariski measure μ and an analytic rigid geometric structure σ of algebraic type and order k. Then there is a G-invariant, conull subset S of M where the action is locally free and such that for every $x \in S$, $v \in T_xGx$ and $r \geq 0$, there is $j_x^{k+r}(X) \in \mathrm{Kill}^{k+r}(\sigma, x)$ with $X_x = v$ and such that $[X, Y^*] = 0$ up to order $k + r - 1$ at x, for every $Y \in \mathfrak{g}$.*

Proof. It suffices to prove that the set of points where the desired conclusion is satisfied is conull with respect to almost every ergodic component of the measure μ. By Proposition 3.8 there is an open subset $U \subset M$ which is conull with respect to μ, invariant under the action of G, and on which this action is locally free. In particular, U is conull with respect to almost every ergodic component of μ. The argument that follows will be applied to any such ergodic component.

Let $r \geq 0$ be a fixed integer. For $x \in U$, let $\mathcal{G}_x^{k+r} = \{j_x^{k+r-1}(X^*) | X \in \mathfrak{g}\}$, where X^* denotes the vector field on U induced by $X \in \mathfrak{g}$. Then $\mathcal{G}^{k+r} = \bigcup_{x\in U} \mathcal{G}_x^{k+r}$ defines a G-invariant analytic vector bundle over U. Moreover, the map

$$\begin{aligned} U \times \mathfrak{g} &\rightarrow \mathcal{G}^{k+r} \\ (x, X) &\mapsto j_x^{k+r-1}(X^*), \end{aligned}$$

defines a G-equivariant analytic trivialization of this bundle, the action of G on $U \times \mathfrak{g}$ being given by $g(x, X) = (gx, \mathrm{Ad}_G(g)(X))$. In particular, the frame bundle $L(\mathcal{G}^{k+r})$ of $\mathcal{G}^{k+r}$ is analytically G-equivariantly equivalent to $U \times \mathrm{Gl}(\mathfrak{g})$ with the G-action on the latter space given by $g(x, A) = (gx, \mathrm{Ad}_G(g)A)$.

For any $j^{k+r}(\varphi) \in L^{(k+r)}(M)$, with $\varphi(0) = x$, define the linear map

$$\begin{aligned} d^{(k+r)}\varphi \colon T_0^{(k+r)}\mathbb{R}^n &\rightarrow T_x^{(k+r)}M \\ j^{k+r-1}(X) &\mapsto j_x^{k+r-1}(d\varphi(X)), \end{aligned}$$

which clearly depends only on $j^{k+r}(\varphi)$. Consider $\mathfrak{g}$ as a (fixed) subspace of $T_0^{(k+r)}\mathbb{R}^n$ (by choosing some subspace of the latter with the same dimension as $\mathfrak{g}$) and define $L^{(k+r)}(U, \mathcal{G}^{k+r})$ as the analytic reduction of $L^{(k+r)}(M)$ over U consisting of $j^{k+r}(\varphi) \in L^{(k+r)}(M)$ such that $\varphi(0) = x \in U$ and $d^{(k+r)}\varphi(\mathfrak{g}) = \mathcal{G}_x^{k+r}$. Let f_{k+r} be the analytic map defined by:

$$\begin{aligned} f_{k+r}\colon L^{(k+r)}(U, \mathcal{G}^{k+r}) &\rightarrow L(\mathcal{G}^{k+r}) \\ j^{k+r}(\varphi) &\mapsto d^{(k+r)}\varphi|_{\mathfrak{g}}. \end{aligned}$$

Then there is an obvious surjective homomorphism from the structure group of $L^{(k+r)}(U, \mathcal{G}^{k+r})$ onto $\mathrm{Gl}(\mathfrak{g})$, the structure group of $L(\mathcal{G}^{k+r})$, so that f_{k+r} is a bundle homomorphism equivariant with respect to such surjection.

Let Q_1 and Q_2 be reductions of $L^{(k+r)}(U, \mathcal{G}^{k+r})$ and $L(\mathcal{G}^{k+r})$, respectively, obtained by applying Proposition 8.2 to the bundle map f_{k+r}. By Proposition 8.3 and the last claim in Proposition 8.1, it may be assumed that Q_2 is equal to $U \times \overline{\mathrm{Ad}_G(G)}^Z$ (except perhaps on a null set), where $\overline{\mathrm{Ad}_G(G)}^Z$ denotes the Zariski closure of $\mathrm{Ad}_G(G)$ in $\mathrm{Gl}(\mathfrak{g})$. Let S_1 be the measurable subset of M consisting of those points $x \in M$ such that a connected component of the fiber of Q_1 at x is mapped by f_{k+r} onto a connected component of the fiber of Q_2 at x. (By the choices made, the set S_1 is conull.) Then for any $x \in S_1 \cap U$ and $g \in G$, there exist $\alpha_x, \alpha_{g,x} \in Q_1$ such that $f_{k+r}(\alpha_x) = (x, e)$ and $f_{k+r}(\alpha_{g,x}) = (x, \mathrm{Ad}_G(g))$. It follows from the definition of f_{k+r} that $\alpha_{g,x} \circ \alpha_x^{-1}$ (considered as an element in $D_x^{(k+r)}(M)$ acting on $T_x^{(k+r)}M$, as described in Section 4) leaves $\mathcal{G}_x^{k+r}$ invariant and acts on it by $\mathrm{Ad}_G(g)$.

Moreover, by Proposition 8.4 there is a G-invariant, conull subset S_2 of M such that $\sigma^r(L^{(k+r)}(M)|_{S_2})$ lies in a single $\mathrm{Gl}^{(k+r)}(n)$-orbit. In particular, if q_0 is chosen in this orbit, then $P = (\sigma^r)^{-1}(q_0)$ is a measurable reduction of $L^{(k+r)}(M)$ over S_2 to the stabilizer of q_0. By the properties of the algebraic hull and since the stabilizer of q_0 is algebraic, there is a conull subset $S_3 \subset M$ and an element $h \in \mathrm{Gl}^{(k+r)}(n)$ such that $Q_1 h \subset P$ over S_3. Then with the above notation, it holds that

$$\sigma^r((\alpha_{g,x} \circ \alpha_x^{-1})\alpha_x h) = \sigma^r((\alpha_{g,x} h) \circ (\alpha_x h)^{-1} \alpha_x h) = \sigma^r(\alpha_{g,x} h) = \sigma^r(\alpha_x h),$$

for every $x \in U \cap S_1 \cap S_2 \cap S_3$, where the last identity follows from the fact that σ^r is constant on P (where both $\alpha_{g,x} h$ and $\alpha_x h$ lie). This identity implies that $\alpha_{g,x} \circ \alpha_x^{-1} \in \mathrm{Aut}^{k+r}(\sigma, x)$.

Let $H\colon D_x^{(k+r)}(M) \rightarrow \mathrm{Gl}(T_x^{(k+r)}M)$ be the representation considered in Section 4. The previous discussion shows that if A_x is the subgroup of $\mathrm{Aut}^{k+r}(\sigma, x)$ consisting of those elements $a \in \mathrm{Aut}^{k+r}(\sigma, x)$ such that $H(a)(\mathcal{G}_x^{k+r}) = \mathcal{G}_x^{k+r}$, then the induced homomorphism $\widetilde{H}: A_x \to \mathrm{Gl}(\mathcal{G}_x^{k+r})$ has image containing $\mathrm{Ad}_G(G)$ (where $\mathcal{G}_x^{k+r}$ is identified with $\mathfrak{g}$ as before). Using Lemma 2.1 within this setup, it follows that, with respect to the representation $h: \mathcal{D}_x^{(k+r)}(M) \to \mathfrak{gl}(T_x^{(k+r)}M)$ described in Section 4, the Lie algebra $\mathrm{Lie}(A_x)$ is contained in $\mathrm{Kill}_0^{k+r}(\sigma, x)$, it satisfies $h(\mathrm{Lie}(A_x))(\mathcal{G}_x^{k+r}) \subset \mathcal{G}_x^{k+r}$, and its induced representation $\widetilde{h}: \mathrm{Lie}(A_x) \to \mathfrak{gl}(\mathcal{G}_x^{k+r})$ has image containing $\mathrm{ad}_G(\mathfrak{g})$. This last property implies that for every $Z \in \mathfrak{g}$ there is an infinitesimal Killing field $j_x^{k+r}(X) \in \mathrm{Kill}_0^{k+r}(\sigma, x)$ such that $j_x^{k+r-1}([X, Y^*]) = j_x^{k+r-1}([Z^*, Y^*])$ for every $Y \in \mathfrak{g}$. We conclude that $j_x^{k+r}(X - Z^*)$ is an infinitesimal Killing field of order $k + r$ that centralizes Y^* up to order $k + r - 1$ at x, for every $Y \in \mathfrak{g}$, and such that $(X - Z^*)_x = -Z_x^*$ which can be chosen to be any previously given element in T_xGx. Note that such infinitesimal Killing fields can be obtained for every $x \in U \cap S_1 \cap S_2 \cap S_3$ which is a G-invariant, conull subset of M.

The infinitesimal Killing fields provided by this result can be extended to local Killing fields for suitable geometric structures. They in turn can be extended to global Killing fields using the results of Amores [1], thus allowing us to complete the proof of Gromov's centralizer theorem. Note that it is explicitly proved in [1] that every analytic local Killing field on a simply connected manifold can be extended to a global Killing field only for geometric structures given by order 1 reductions of finite type. However, the techniques of [1] can be applied without essential modifications to prove that the same extension property holds true for the geometric structures considered in the next theorem. In fact, a remark of this sort is made in [1] regarding connections on the first order frame bundle. We will use this observation in the proof of Gromov's centralizer theorem.

THEOREM 9.2 (Gromov's centralizer theorem). *Let M be a connected analytic manifold endowed with an analytic rigid structure σ of one of the following types: a connection, a parallelism on some frame bundle, or a geometric structure of order 1 defined by a reduction of $L(M)$ to an algebraic group.*

Let G be a connected, noncompact, simple Lie group acting analytically on M, preserving both σ and a finite Zariski measure.

Let $\mathcal{G}$ be the Lie algebra of Killing vector fields on the universal cover $\widetilde{M}$ induced by the action of the universal cover $\widetilde{G}$ of G. If $\mathcal{V}$ denotes the space of analytic Killing vector fields on $\widetilde{M}$ that centralize $\mathcal{G}$, then:

1. *$\mathcal{V}$ is $\pi_1(M)$-invariant,*

2. *$\mathcal{V}$ is finite dimensional,*

3. *there is an open, conull subset $\widetilde{U}$ of $\widetilde{M}$, invariant under both G and $\pi_1(M)$, on which G acts locally freely and such that $\mathrm{ev}_x(\mathcal{V}) \supset T_xGx$ for every $x \in \widetilde{U}$.*

Here, for a point $x \in M$ and a vector field X defined in a neighborhood of x, $\mathrm{ev}_x(X) = X_x$ is the evaluation map.

Proof. Part (1) of the theorem follows from the fact that the geometric structure σ lifted to $\widetilde{M}$ (which will be denoted with the same symbol) is $\pi_1(M)$-invariant and the actions of $\widetilde{G}$ and $\pi_1(M)$ commute. Furthermore, because σ is rigid, every $X \in \mathcal{V}$ (being analytic) is determined by its jet at any point of a high enough but fixed order, from which part (2) follows. It remains to prove (3).

Let S be a conull subset of M provided by Lemma 9.1. Then $\widetilde{S} = \pi^{-1}(S)$ satisfies the conclusion of Lemma 9.1 for the action of $\widetilde{G}$ on $\widetilde{M}$, where $\pi: \widetilde{M} \to M$ is the natural projection. Fix $x \in \widetilde{S}$ and $v \in T_x\widetilde{G}x$. If σ is of order k, then for every $r \geq 0$, the set $\mathcal{V}_x^{k+r}$ of infinitesimal Killing fields $j_x^{k+r}(X) \in \mathrm{Kill}^{k+r}(\sigma, x)$ such that $X_x = v$ and $[X, Y^*] = 0$ up to order $k + r - 1$ at x is a nonempty affine subspace of $\mathrm{Kill}^{k+r}(\sigma, x)$. Moreover, $\pi_{k+r}^{k+r+1}(\mathcal{V}_x^{k+r+1}) \subset \mathcal{V}_x^{k+r}$ for every $r \geq 0$, where π_{k+r}^{k+r+1} denotes the natural jet projection. Thus, as σ is $(k+r_0)$-rigid for some integer r_0, $(\mathcal{V}_x^{k+r})_{r \geq r_0}$ is a descending sequence of affine spaces with respect to the jet projections. Hence, there is an integer $r_1(x)$ after which this sequence stabilizes.

Propositions 6.10, 6.11, and 7.13 imply that there is an integer $r_2(x)$ such that any infinitesimal Killing field at x of order $\geq k + r_2(x)$ extends uniquely to a local Killing field. Let $r(x) = \max(r_1(x), r_2(x), r_0)$ and for every $r \geq r(x)$ choose $j_x^{k+r}(X^r) \in \mathcal{V}_x^{k+r}$. Since the projection $\pi_{k+r}^{k+r+1}: \mathcal{V}_x^{k+r+1} \to \mathcal{V}_x^{k+r}$ is bijective we can proceed inductively in our selection to assume that $j_x^{k+r}(X^s) = j_x^{k+r}(X^r)$ whenever $s \geq r \geq r(x)$. Let X be a Killing field defined in a neighborhood of x such that $j_x^{k+r(x)}(X) = j_x^{k+r(x)}(X^{r(x)})$. In particular, $j_x^{k+r}(X)$ is an infinitesimal Killing field for every $r \geq r(x)$, and by the choice of $j_x^{k+r}(X^r)$ together with $(k + r_0)$-rigidity if follows that $j_x^{k+r}(X) = j_x^{k+r}(X^r)$ for every $r \geq r(x)$. Therefore, by analyticity, X is a local Killing field such that $X_x = v$ and such that it centralizes $\mathcal{G}$ in a neighborhood of x. By the main results in [1], X can be extended to a global Killing field on M satisfying $X_x = v$ and (again by analyticity) centralizing $\mathcal{G}$ on all of $\widetilde{M}$.

The above discussion proves that the inclusion considered in part (3) is satisfied on a G-invariant conull subset of $\widetilde{M}$. It remains to show that such set may be assumed to be open.

Let $\widetilde{U}_0$ be a $\widetilde{G}$-invariant, conull, dense open subset of $\widetilde{M}$ on which the action of $\widetilde{G}$ is locally free (as provided by Proposition 3.8), and let

$\widetilde{W}_0$ be a non-null connected component of $\widetilde{U}_0$. (Note that we can, and do, assume that $\widetilde{W}_0$ is $\widetilde{G}$-invariant.) Consider the natural evaluation map $\mathrm{ev}\colon \widetilde{W}_0 \times \mathcal{V} \to T\widetilde{M}$. Since ev is analytic, Corollary 3.6 implies the existence of a $\widetilde{G}$-invariant, conull, dense, open subset $\widetilde{U}_1 \subset \widetilde{W}_0$ on which $\mathrm{rank}(\mathrm{ev}_x)$ is maximal. Let $\widetilde{W}_1$ be a non-null connected component of $\widetilde{U}_1$. By the maximality of $\mathrm{rank}(\mathrm{ev}_x)$ it follows that $E = \bigcup_{x\in\widetilde{W}_1} \mathrm{ev}_x(\mathcal{V})$ defines an analytic vector subbundle of $T\widetilde{W}_1$.

Further, we know that $T\mathcal{O}|_{\widetilde{W}_1} = \bigcup_{x\in\widetilde{W}_1} T_xGx$ defines an analytic vector bundle over $\widetilde{W}_1$. Let φ be the restriction to $T\mathcal{O}|_{\widetilde{W}_1}$ of the natural projection $T\widetilde{W}_1 \to T\widetilde{W}_1/E$. Then φ is an analytic map of vector bundles, which satisfies that, for $x \in \widetilde{W}_1$, $\mathrm{ev}_x(\mathcal{V}) \supset T_xGx$ if and only if $\varphi_x = 0$. This condition then defines an analytic subset X_1 of $\widetilde{W}_1$. If X_1 is a proper subset of $\widetilde{W}_1$, then it is null and the inclusion in part (3) is satisfied only on a null subset of $\widetilde{W}_1$, which is impossible since $\widetilde{W}_1$ is not null. Hence $X_1 = \widetilde{W}_1$ and so the inclusion in (3) is satisfied on all $\widetilde{W}_1$. Our argument has discarded only null open components and null analytic sets from $\widetilde{U}_0$ to obtain a $\widetilde{G}$-invariant open set $\widetilde{U}$ on which condition (3) is satisfied. In particular, $\widetilde{U}$ is conull in $\widetilde{U}_0$ and it is also easily seen to be $\pi_1(M)$-invariant. Since $\widetilde{U}_0$ is itself conull we obtain the desired conclusion.

Remark. In the above theorem, if the G-invariant Zariski measure on M is positive on open sets, as is the case for a smooth measure, then the set $\widetilde{U}$ in (3) can be assumed to be dense. This is so because in the last paragraph of the above proof we only discard null analytic sets from the open set $\widetilde{U}_0$ coming from Proposition 3.8.

10. Gromov's representation and applications

In this section, M denotes a connected analytic manifold and σ an analytic geometric structure on M of one of the types listed in the statement of Gromov's centralizer theorem. The manifold M is not required to be compact. We also assume that G is a connected, non-compact, simple Lie group which acts (analytically) on M preserving σ and a finite Zariski measure.

The following theorem, for M a compact manifold, is Theorem 1.5 in Zimmer [20]. The proof of our result follows the arguments of [20] with the use of Theorem 9.2.

THEOREM 10.1. *There is a representation $\rho\colon \pi_1(M) \to \mathrm{Gl}(q)$ such that the Zariski closure of $\rho(\pi_1(M))$ contains a subgroup locally isomorphic to G.*

A very important consequence of Gromov's centralizer theorem is the fact that suitable actions of simple Lie groups are topologically engaging. This fact (for compact manifolds) is essentially contained in the statement of Corollary 4.5 of Zimmer [20]. A detailed proof can also be found in [2]. Several results from [18] show that topologically engaging actions are indeed useful when studying actions of semisimple groups. None of the results of [18] require that the manifold acted upon be compact, a fact which emphasizes the need to prove topological engagement for noncompact manifolds. It follows from Gromov's centralizer theorem and the arguments from [2] that the actions considered in this section are topologically engaging. Most applications of topological engagement can be carried out with such condition satisfied on a sufficiently large set, as the one considered in the following result.

THEOREM 10.2. *Suppose that G has finite center and finite fundamental group. Then the action of G on M is topologically engaging on a conull, open subset of M. Moreover, there is a $\widetilde{G}$- and $\pi_1(M)$-invariant, conull, open subset $\widetilde{U}$ of $\widetilde{M}$ such that each of its $\widetilde{G}$-orbits is closed in $\widetilde{U}$. If the measure is smooth, then the open sets in both M and $\widetilde{M}$ where the topological engagement condition is satisfied can be assumed to be dense.*

The following result, under the assumption that the manifold is compact, is Theorem A of Spatzier and Zimmer [13]. The invariant geometric structure is needed to ensure that the action is topologically engaging, based on the previously known Gromov's centralizer theorem for compact manifolds. This is the only point in [13] where compactness is used. Hence for the following result in the finite volume case Theorem 10.2 allows us to simply apply in a straightforward way the arguments found in [13].

THEOREM 10.3. *Suppose that G has real rank at least 2, finite center and finite fundamental group. Then $\pi_1(M)$ is not isomorphic to the fundamental group of a complete Riemannian manifold with negative sectional curvature bounded away from 0 and $-\infty$.*

Like the previous theorem, the following results can be obtained using arguments similar to those of Zimmer [19] or [20], using Theorem 10.2 to ensure topological engagement in the finite volume case.

THEOREM 10.4. *Assume that G has real rank at least 2, finite center and finite fundamental group. Suppose that $\pi_1(M)$ admits a faithful linear representation $\rho: \pi_1(M) \to \mathrm{Gl}(k, \mathbb{C})$ for some k such that:*

1. *$\rho(\pi_1(M))$ is discrete; or*
2. *$\rho(\pi_1(M)) \subset \mathrm{Gl}(k, \overline{\mathbb{Q}})$.*

Then $\pi_1(M)$ contains a lattice in a linear Lie group L, where L contains a group locally isomorphic to G.

Thus the fundamental group of M is not isomorphic to a lattice in a Lie group with real rank strictly smaller than that of G. In particular, it is not isomorphic to a discrete subgroup of a Lie group of real rank one.

THEOREM 10.5. *Fix G as above with real rank at least 2, finite center and finite fundamental group and let m be a fixed positive integer. Then there is a finitely generated group Λ that satisfies the following conditions:*

1. *if $\dim M \leq m$, then there is no isogeny (surjection with finite kernel) $\pi_1(M) \to \Lambda$; and*
2. *there exists a compact analytic manifold N with an isogeny $\pi_1(N) \to \Lambda$ such that there is a real analytic action of G on N preserving an analytic pseudo-Riemannian metric.*

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