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Parallelisms, prolongations of Lie algebras and rigid geometric structures

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Abstract. For geometric structures of type Q, we prove that being rigid depends only on the stabilizers for the action on Q. We also prove that to any rigid structure we can associate a "natural" parallelism. Moreover, if the rigid structure is analytic, then the parallelism can be taken to be analytic as well. This implies an extension theorem for infinitesimal Killing fields. As an application we obtain Gromov's centralizer theorem for arbitrary analytic rigid unimodular structures.

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1. Introduction

A geometric structure of order k and type Q on a manifold is given by an equivariant map from the k-th order frame bundle of the manifold into Q. Here Q is a $\mathrm{Gl}^{(k)}(n)$-space, where $\mathrm{Gl}^{(k)}(n)$ is the structure group of the k-th order frame bundles on n-dimensional manifolds. If $Q = \mathrm{Gl}^{(k)}(n)/H$, where H is a closed subgroup of $\mathrm{Gl}^{(k)}(n)$, then the geometric structure is called an H-structure. Our main interest is on rigid geometric structures, defined in Gromov [6] as those whose automorphisms are determined by some jet of fixed order.

For H-structures of order 1 (reductions of linear frame bundles) we proved in [2] that being rigid is equivalent to being of finite type (as defined in Kobayashi [7]). In [3] we developed a notion of H-structure of finite type for reductions of higher order frame bundles. This was achieved by generalizing the constructions of prolongations of Lie algebras and reductions, which are well known for linear frame bundles (cf. Kobayashi [7]), to reductions of higher order frame bundles and their structure groups. In particular, in [3] an H-structure of finite type was defined as one for which a suitable prolongation of the Lie algebra of H vanishes, and it was shown there that associated to such finite type H-structure there is a finite tower

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of principal bundles and a connection on the top bundle so that an automorphism of the H-structure induces an automorphism of the connection; thus extending the corresponding results for H-structures of order 1 found in [7]. It was also proved in [3] that, for H-structures of any order, the conditions of being rigid and being of finite type are equivalent.

These results obtained in [2], [3] yield two important properties of rigid H-structures. One property is that an H-structure is rigid if and only if some prolongation of the Lie algebra of H vanishes. Thus, for H-structures, the condition of being rigid depends only on the type, i.e., depends only on $\mathrm{Gl}^{(k)}(n)/H$ as a $\mathrm{Gl}^{(k)}(n)$-space. The other property is that the parallelism given by the connection on the top bundle of the tower of principal bundles associated to an H-structure of finite type is in a certain sense natural because it determines the automorphisms of the H-structure.

The main goal of this work is to extend these properties of rigid H-structures to rigid geometric structures of arbitrary type Q, not necessarily a homogeneous $\mathrm{Gl}^{(k)}(n)$-space.

It will be shown in Section 4 that, for arbitrary geometric structures, the condition of being rigid can still be described in terms of prolongations of Lie algebras, as defined in [3]. In fact, Theorem 1 shows that a geometric structure σ of order k and type Q is rigid if and only if the image of σ is contained in an open subset of Q on which the stabilizers of points (for the $\mathrm{Gl}^{(k)}(n)$-action on Q) have Lie algebras with vanishing prolongations. In the case of H-structures, this is just the equivalence between rigid and finite type; thus, in general, rigid geometric structures are essentially finite type structures whose type Q is not necessarily a homogeneous $\mathrm{Gl}^{(k)}(n)$-space. Moreover, for geometric structures of order k and type Q, Theorem 1 describes rigidity in terms of the $\mathrm{Gl}^{(k)}(n)$-action on Q, so that being rigid depends only on the type Q.

Section 5 discusses the problem of associating "natural" parallelisms to geometric structures. To this end we define a framed rigid geometric structure σ as one for which there is a parallelism on some frame bundle so that automorphisms of σ induce automorphisms of the parallelism (cf. Definition 5). Theorem 2 shows that every rigid geometric structure is also framed rigid, and so a smooth rigid structure has an associated smooth parallelism. This result is also proved in the analytic category.

D'Ambra-Gromov [4] introduced a notion of framed rigid geometric structure based on associating a parallelism by a "canonical procedure." Their definition differs from ours in that our parallelisms may not be unique. Nonetheless, Lemma 3 shows that, in the analytic category, suitable Killing fields for any parallelism associated to a framed rigid structure σ (as in our definition) are Killing fields of σ. Thus, for our analytic framed rigid structures, any associated parallelism determines the geometric structure to a certain degree, and so it is in a sense natural. Our definition of framed rigidity has the further advantage of making it easier to deal with Killing fields. This allows us to use Theorem 2 to obtain an extension theorem for infinitesimal Killing fields for analytic rigid structures (Theorem 3). As an application of this we obtain in Section 6 Gromov's centralizer theorem (Theorem 4) and a Gromov's representation (Theorem 5) for analytic actions of simple groups on finite volume manifolds preserving an arbitrary analytic rigid unimodular structure. Such results were previously proved in Gromov [6] only for compact manifolds and in [2], [3] for finite volume manifolds only when the structure is an H-structure.

2. Preliminaries on jet bundles

The results and constructions in this section are presented in the smooth category, but all of them can also be considered in the analytic category.

Two smooth maps $f, g : M \to Q$ between smooth manifolds have the same r-jet at $x \in M$ if $f(x) = g(x)$ and they have the same partial derivatives up to order r at x (in terms of local coordinate systems around x and $f(x)$). The r-jet determined by f at x is usually denoted by $j^r_x(f)$. In all the jet spaces considered in this work and for $l \geq k$ we will denote with π^l_k the natural jet projection sending $j^l_x(f)$ into $j^k_x(f)$. Usually the context will determine the jet spaces on which π^l_k is being considered.

Let $J^r_n(Q)$ denote the smooth manifold of r-jets at the origin $0 \in \mathbb{R}^n$ of smooth maps $f : \mathbb{R}^n \to Q$. For simplicity in notation, the r-jet of a smooth map $f : \mathbb{R}^n \to Q$ at the origin $0 \in \mathbb{R}^n$ will be denoted by $j^r(f)$ instead of $j^r_0(f)$.

Note also that if Q is a Lie group, then $J^r_n(Q)$ inherits a group structure defined by $j^r(g_1)j^r(g_2) = j^r(g_1g_2)$.

Let $\mathrm{Gl}^{(k)}(n)$ denote the group of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ that fix 0. If $S_i(\mathbb{R}^n; \mathbb{R}^n)$ denotes the space of $\mathbb{R}^n$-valued i-multilinear transformations on $\mathbb{R}^n$, then as a manifold we have

$$\mathrm{Gl}^{(k)}(n) = \{(A, L_2, \ldots, L_k) \mid A \in \mathrm{Gl}(n), L_i \in S_i(\mathbb{R}^n; \mathbb{R}^n) \text{ for every } i \geq 2\},$$

and is in fact a Lie group. Note that $\mathrm{Gl}^{(1)}(n)$ is the general linear group $\mathrm{Gl}(n)$ and for any pair of integers $l \geq k$ there is a canonical homomorphism $\pi^l_k \colon \mathrm{Gl}^{(l)}(n) \to \mathrm{Gl}^{(k)}(n)$. The kernel of π^k_{k-1} will be denoted by N^k and is given by the set of $\mathbb{R}^n$-valued k-multilinear transformations on $\mathbb{R}^n$. Also, we will denote with $\mathfrak{gl}^{(k)}(n)$ the Lie algebra of $\mathrm{Gl}^{(k)}(n)$, and so we have in particular that $\mathfrak{gl}^{(1)}(n) = \mathfrak{gl}(n)$ as Lie algebras. $\mathfrak{gl}^{(k)}(n)$ can be described as the Lie algebra of k-jets at 0 of vector fields on $\mathbb{R}^n$ that vanish at 0. Hence, as a vector space we have

$$\mathfrak{gl}^{(k)}(n) = \{(L_1, \ldots, L_k) \mid L_i \in S_i(\mathbb{R}^n; \mathbb{R}^n) \text{ for every } i \geq 1\},$$

from which it follows that N^k is a subspace of $\mathfrak{gl}^{(k)}(n)$.

Consider the map $a : \mathrm{Gl}^{(k+r)}(n) \to J^r_n(\mathrm{Gl}^{(k)}(n))$ defined as follows. If $g \in \mathrm{Gl}^{(k+r)}(n)$ is of the form $g = j^{(k+r)}(f)$, let $f_k : \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$ be the map given by

$$f_k(x) = j^k(\tau_{-x} \circ f \circ \tau_{f^{-1}(x)}),$$

where $\tau_v(y) = y + v$ is the translation by v in $\mathbb{R}^n$, and set

$$a(g) = j^r(f_k).$$

This map a satisfies

$$a(g_1 g_2) = a(g_1)(a(g_2) \circ \pi^{k+r}_r(g_1^{-1})),$$

where '$\circ$' denotes the operation given by $j^r(f) \circ j^r(\varphi) = j^r(f \circ \varphi)$. From this it follows that if $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ is the semi-direct product with group multiplication $(g, h)(g', h') = (gg', h(h' \circ g^{-1}))$, then the map $(\pi^{k+r}_r, a) : \mathrm{Gl}^{(k+r)}(n) \to \mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ is a homomorphism of Lie groups.

In this work, for a given transformation $L \in N^{k+r}$ we will denote with $L_{(r)}$ the N^k-valued r-multilinear symmetric transformation over $\mathbb{R}^n$ given by the assignment $(v_1, \dots, v_r) \mapsto L(v_1, \dots, v_r, \ , \dots, \ )$.

Lemma 1. *The homomorphism*

$$(\pi^{k+r}_r, a) : \mathrm{Gl}^{(k+r)}(n) \to \mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$$

maps N^{k+r} into $J^r_n(N^k)$.

Proof. Each $g \in N^{k+r}$ can be represented as $g = j^{k+r}(\varphi)$, where $\varphi(y) = y + (1/(k+r)!)L(y, \dots, y)$, L being a symmetric $\mathbb{R}^n$-valued $(k+r)$-multilinear map on $\mathbb{R}^n$. Correspondingly, $a(g) = j^r(\bar{L})$, where $\bar{L} : \mathbb{R}^n \to N^k \subset \mathrm{Gl}^{(k)}(n)$ is given by $\bar{L}(x) = (I, 0, \dots, 0, (1/r!)L_{(r)}(x, \dots, x))$.

If Q is a manifold that admits a smooth action of $\mathrm{Gl}^{(k)}(n)$ (on the left), then $J^r_n(Q)$ admits a smooth action of $\mathrm{Gl}^{(k+r)}(n)$ (on the left) which is called the r-prolongation of the action on Q.

This r-prolongation is constructed as follows. There is a natural left action of the Lie group $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ on $J^r_n(Q)$ given by $(g, h)q = h \cdot (q \circ g^{-1})$, where '$\circ$' is defined as before and '$\cdot$' is given by $j^r(f_1) \cdot j^r(f_2) = j^r(f_1 f_2)$. This induces, with the help of (π^{k+r}_r, a), a canonical action $\mathrm{Gl}^{(k+r)}(n) \times J^r_n(Q) \to J^r_n(Q)$ given by $gq = a(g) \cdot (q \circ \pi^{k+r}_r(g^{-1}))$.

Finally, there is a natural smooth embedding $i \colon J^{r+1}_n(Q) \to J^1_n(J^r_n(Q))$ given by $j^{r+1}(f) \mapsto j^1(\tilde{f})$, where $\tilde{f}(x) = j^r(f \circ \tau_x)$ for $f \colon \mathbb{R}^n \to Q$. The above discussion defines a smooth action of $\mathrm{Gl}^{(k+r+1)}(n)$ on $J^{r+1}_n(Q)$, and this same construction applied to the action of $\mathrm{Gl}^{(k+r)}(n)$ on $J^r_n(Q)$ defines a smooth action of $\mathrm{Gl}^{(k+r+1)}(n)$ on $J^1_n(J^r_n(Q))$. It is straightforward to show from the definitions that i is $\mathrm{Gl}^{(k+r+1)}(n)$-equivariant with respect to such actions. In words, the $(r+1)$-prolongation of the action of $\mathrm{Gl}^{(k)}(n)$ on Q and the 1-prolongation of the action of $\mathrm{Gl}^{(k+r)}(n)$ on $J^r_n(Q)$ coincide on the subset $J^{r+1}_n(Q)$ of $J^1_n(J^r_n(Q))$.

3. Automorphisms of geometric structures

This section contains some basic definitions and properties of geometric structures and of their automorphisms. As before, the statements are given in the smooth category, but remain valid without change in the analytic category. In what follows, $L^{(k)}(M)$ denotes the k-th order frame bundle of a manifold M, which is a $\mathrm{Gl}^{(k)}(n)$-principal bundle if M is n-dimensional. If f is a diffeomorphism between manifolds, then $f_{(k)}$ denotes the canonical diffeomorphism that it induces between the corresponding k-th order frame bundles. If X is a vector field on a manifold M with local flow φ_t, then $X_{(k)}$ denotes the vector field on $L^{(k)}(M)$ whose local flow is $(\varphi_t)_{(k)}$. Further details and properties can be found in [3] and in Kolář, Michor and Slovák [8].

Definition 1. *Let Q be a smooth manifold on which $\mathrm{Gl}^{(k)}(n)$ acts smoothly on the left. A smooth geometric structure of order k and type Q on an n-dimensional smooth manifold M is a smooth $\mathrm{Gl}^{(k)}(n)$-equivariant map $L^{(k)}(M) \to Q$.*

Let σ be a smooth geometric structure of order k and type Q over M. Then a (local) diffeomorphism φ of M is called a (local) automorphism if it (locally) satisfies $\sigma \circ \varphi_{(k)} = \sigma$. Also, for every $r \geq 0$, the jet $j_x^{k+r}(\varphi)$ of a local diffeomorphism φ is called an infinitesimal automorphism of order $k+r$ if $\sigma \circ \varphi_{(k)} = \sigma$ is satisfied up to order r at some (and hence every) point in the fiber of $L^{(k)}(M)$ over x. Let $\mathrm{Aut}(M,\sigma)$, $\mathrm{Aut}^{\mathrm{loc}}(M,\sigma)$ and $\mathrm{Aut}^{k+r}(\sigma,x,y)$ denote the group, pseudogroup and space of global, local and infinitesimal (of order $k+r$) automorphisms of σ, respectively, where for the latter we consider $(k+r)$-jets at x of local diffeomorphisms of M that map x to y. We also denote $\mathrm{Aut}^{k+r}(\sigma,x,x) = \mathrm{Aut}^{k+r}(\sigma,x)$ which is easily seen to be a Lie group.

For every non-negative integer r and for any given smooth geometric structure of order k and type Q on an n-dimensional smooth manifold M, there is a smooth geometric structure of order $k+r$ and type $J_n^r(Q)$ on M, called its r-th order prolongation. If $\sigma : L^{(k)}(M) \to Q$ is the $\mathrm{Gl}^{(k)}(n)$-equivariant map that defines a smooth geometric structure on M, then the r-th order prolongation of σ is the geometric structure whose associated $\mathrm{Gl}^{(k+r)}(n)$-equivariant map is given by

$$\begin{aligned} \sigma^r : L^{(k+r)}(M) &\to J_n^r(Q) \\ j^{k+r}(\varphi) &\mapsto j^r(\sigma(j^k(\varphi \circ \tau_\bullet))) \end{aligned}$$

where $\sigma(j^k(\varphi \circ \tau_\bullet))$ denotes the map $v \in \mathbb{R}^n \mapsto \sigma(j^k(\varphi \circ \tau_v)) \in Q$. Note that $\sigma^0 = \sigma$ for any geometric structure σ. Also note that $j_x^{k+r}(\varphi) \in \mathrm{Aut}^{k+r}(\sigma,x,y)$ if and only if $\sigma^r \circ \varphi_{(k+r)} = \sigma^r$ in the fiber of $L^{(k+r)}(M)$ over x.

A Killing field for a smooth geometric structure σ on M is a smooth vector field on M whose local flow acts on M by local automorphisms of σ.

The space of Killing fields and local Killing fields of a geometric structure σ are denoted by $\mathrm{Kill}(\sigma)$ and $\mathrm{Kill}^{\mathrm{loc}}(\sigma)$, respectively. It follows from the standard relation between a vector field and its local flow that a vector field X on M is a Killing field for σ if and only if $d\sigma_\alpha(X_{(k)}) = 0$ for every $\alpha \in L^{(k)}(M)$. Also, an infinitesimal Killing field of order $k+r$ at x for σ is a $(k+r)$-jet $j_x^{k+r}(X)$ of a germ at x of a vector field X so that $d\sigma_\alpha^r(X_{(k+r)}) = 0$ for every $\alpha \in L^{(k+r)}(M)$ in the fiber over x. We denote with $\mathrm{Kill}^{k+r}(\sigma, x)$ the space of infinitesimal Killing fields for σ of order $k+r$, and also denote with $\mathrm{Kill}_0^{k+r}(\sigma, x)$ the subspace consisting of those vanishing at x.

Let $j_x^{k+r}(X)$ be a $(k+r)$-jet of a vector field on M. Then a straightforward computation shows that $j_x^{k+r}(X) \in \mathrm{Kill}^{k+r}(\sigma, x)$ if and only if the local flow φ_t of X satisfies $j_x^{k+r}(\varphi_t) \in \mathrm{Aut}^{k+r}(\sigma, x, \varphi_t(x))$, for every t in a neighborhood of 0. Moreover, if $j_x^{k+r}(X) \in \mathrm{Kill}_0^{k+r}(\sigma, x)$, then this last property holds for every $t \in \mathbb{R}$. From this it follows that $\mathrm{Kill}_0^{k+r}(\sigma, x)$ is the Lie algebra of $\mathrm{Aut}^{k+r}(\sigma, x)$.

Recall that if $Q = \mathrm{Gl}^{(k)}(n)/H$ for some closed subgroup H of $\mathrm{Gl}^{(k)}(n)$, endowed with the natural left action of $\mathrm{Gl}^{(k)}(n)$, then there is a natural correspondence between geometric structures of order k and type Q, H-reductions of $L^{(k)}(M)$ and sections of the bundle $L^{(k)}(M)/H \to M$. Given this correspondence it is a simple exercise to prove the following result.

Proposition 1. *Let H be a closed subgroup of $\mathrm{Gl}^{(k)}(n)$ and σ a smooth geometric structure of order k and type $\mathrm{Gl}^{(k)}(n)/H$ over an n-dimensional smooth manifold M. Let $\lambda\colon M \to L^{(k)}(M)/H$ be the section and P the H-reduction associated to σ. If φ is a (local) diffeomorphism of M, then the following conditions are equivalent:*

1. *φ is a (local) automorphism of σ,*
2. *$\lambda \circ \varphi = \hat{\varphi} \circ \lambda$ (locally),*
3. *$\varphi_{(k)}(P) = P$ (locally),*

where $\hat{\varphi}$ denotes the (local) diffeomorphism of $L^{(k)}(M)/H$ induced by $\varphi_{(k)}$.

A similar characterization of infinitesimal automorphisms will be required. The following notion of contact between submanifolds of a given manifold is introduced for that purpose.

Definition 2. *Two m-dimensional submanifolds N_1, N_2 of a manifold M have r-th order contact at a point x if there exist local parametrizations $\varphi_i\colon U_i \to N_i$ defined on open neighborhoods U_i of 0 in $\mathbb{R}^m$ with $\varphi_i(0) = x$, for $i = 1, 2$, and such that $j^r(\varphi_1) = j^r(\varphi_2)$.*

Lemma 2. *Let $\pi\colon P \to M$ be a principal G-bundle and H a closed subgroup of G. For $\sigma\colon P \to G/H$ a smooth G-equivariant map, let $\lambda\colon M \to P/H$ be the section and Q the H-reduction associated to σ. If φ_1 is a bundle diffeomorphism of P covering a diffeomorphism φ of M and $\hat{\varphi}$ is the induced diffeomorphism of P/H, then for every $r \geq 0$ and $x \in M$ the following conditions are equivalent:*

1. *$j^r_\alpha(\sigma \circ \varphi_1) = j^r_\alpha(\sigma)$ for every $\alpha \in \pi^{-1}(x)$,*
2. *$j^r_x(\lambda \circ \varphi) = j^r_x(\hat{\varphi} \circ \lambda)$,*
3. *$\varphi_1(Q)$ and Q have contact of order r at every $\alpha \in \pi^{-1}(\varphi(x)) \cap Q$.*

Proof. It is easy to see that $Q = \sigma^{-1}(eH)$, and also that the natural smooth map

$$\begin{aligned} P \times G/H &\longrightarrow P/H \\ (\alpha, gH) &\longmapsto \alpha * gH = (\alpha g)H \end{aligned}$$

satisfies $\alpha * \sigma(\alpha) = \lambda(\pi(\alpha))$, for every $\alpha \in P$.

If (1) holds, then the smooth maps $P \to P/H$ given by:

$$\begin{aligned} \alpha &\mapsto \lambda \circ \varphi \circ \pi(\alpha) = \lambda \circ \pi \circ \varphi_1(\alpha) = \varphi_1(\alpha) * \sigma(\varphi_1(\alpha)) \\ \alpha &\mapsto \hat{\varphi} \circ \lambda \circ \pi(\alpha) = \hat{\varphi}(\alpha * \sigma(\alpha)) = \varphi_1(\alpha) * \sigma(\alpha) \end{aligned}$$

have the same r-jet at every $\alpha \in \pi^{-1}(x)$. This implies (2) because π is a submersion.

If (2) holds, then, since $\hat{\varphi} \circ \lambda$ and $\lambda \circ \varphi$ are immersions, $\hat{\varphi} \circ \lambda(M)$ and $\lambda \circ \varphi(M)$ have contact of order r at $\hat{\varphi} \circ \lambda(x) = \lambda \circ \varphi(x)$. On the other hand, if $\pi_0 \colon P \to P/H$ denotes the natural projection, then it is easy to check that $Q = \pi_0^{-1}(\lambda(M)) = \pi_0^{-1}(\lambda \circ \varphi(M))$. Now

$$\begin{aligned} \varphi_1(Q) &= \varphi_1(\pi_0^{-1}(\lambda(M))) \\ &= (\pi_0 \circ \varphi_1^{-1})^{-1}(\lambda(M)) \\ &= (\hat{\varphi}^{-1} \circ \pi_0)^{-1}(\lambda(M)) \\ &= \pi_0^{-1}(\hat{\varphi} \circ \lambda(M)), \end{aligned}$$

and thus (3) is a consequence of the following easy to prove property.

Claim. If $F \colon Z \to Y$ is a submersion and Y_1, Y_2 are submanifolds of Y that have contact of order r at y, then $F^{-1}(Y_1), F^{-1}(Y_2)$ have contact of order r at every $z \in F^{-1}(y)$.

Finally, assume that (3) holds. If $\alpha_0 \in \pi^{-1}(x) \cap Q$, then (3) implies that there are local parametrizations $f_1, f_2 \colon \mathbb{R}^k \to Q$ with $f_1(0) = \alpha_0$, $f_2(0) = \varphi_1(\alpha_0)$ and such that $j^r(f_2) = j^r(\varphi_1 \circ f_1)$. It follows from this that the maps $\mathbb{R}^k \times G \to G/H$ given by

$$\begin{aligned} (v, g) &\mapsto \sigma \circ \varphi_1(f_1(v)g) = g^{-1}\sigma(\varphi_1(f_1(v))) \\ (v, g) &\mapsto \sigma(f_1(v)g) = g^{-1}\sigma(f_1(v)) = g^{-1}H = g^{-1}\sigma(f_2(v)) \end{aligned}$$

have the same r-jet at $(0, g)$, for every $g \in G$. Since $(v, g) \mapsto f_1(v)g$ is a submersion, it follows that $j^r_{\alpha_0 g}(\sigma \circ \varphi_1) = j^r_{\alpha_0 g}(\sigma)$, for every $g \in G$, which in turn implies (1).

A consequence of this proposition is the following characterization of infinitesimal automorphisms of geometric structures of type $\mathrm{Gl}^{(k)}(n)/H$.

Proposition 2. *Let H be a closed subgroup of $\mathrm{Gl}^{(k)}(n)$ and σ a smooth geometric structure of order k and type $\mathrm{Gl}^{(k)}(n)/H$ over an n-dimensional smooth manifold M. Let $\lambda\colon M \to L^{(k)}(M)/H$ be the section, and P be the H-reduction, associated to σ. If φ is a local diffeomorphism of M that maps x to y, and $\hat{\varphi}$ denotes the local diffeomorphism of $L^{(k)}(M)/H$ induced by $\varphi_{(k)}$, then the following conditions are equivalent:*

1. $j^{k+r}_x(\varphi) \in \mathrm{Aut}^{k+r}(\sigma, x, y)$,
2. $j^r_x(\lambda \circ \varphi) = j^r_x(\hat{\varphi} \circ \lambda)$,
3. *$\varphi_{(k)}(P)$ and P have contact of order r at every $\alpha \in P$ in the fiber over y.*

4. Characterizing rigid geometric structures

The following definition of rigid geometric structure is due to Gromov [6].

Definition 3. *Let r be a non-negative integer. A smooth geometric structure σ of order k on a smooth manifold M is called r-rigid if, for every $x \in M$, the canonical jet projection $\pi^{k+r+1}_{k+r}\colon \mathrm{Aut}^{k+r+1}(\sigma, x) \to \mathrm{Aut}^{k+r}(\sigma, x)$ is injective. Also, σ is called Killing r-rigid if, for every $x \in M$, the projection*

$$\pi^{k+r+1}_{k+r}\colon \mathrm{Kill}^{k+r+1}_0(\sigma, x) \to \mathrm{Kill}^{k+r}_0(\sigma, x)$$

is injective.

It is well known that if a geometric structure is r-rigid, then it is $(r+1)$-rigid as well (see [2,6]).

The proof of the following proposition is given in [2]. It says that in order to determine whether a geometric structure is rigid or not, it suffices to study the stabilizers for the actions of the groups N^{k+r}.

Proposition 3. *Let $\sigma : L^{(k)}(M) \to Q$ be the $\mathrm{Gl}^{(k)}(n)$-equivariant map defining a smooth geometric structure of order k and type Q on an n-dimensional smooth manifold M. Consider the action of N^{k+r+1} on $J^{r+1}_n(Q)$ induced by the $(r+1)$-prolongation of the action of $\mathrm{Gl}^{(k)}(n)$ on Q. Then σ is r-rigid, for some $r \geq 0$, if and only if the action of N^{k+r+1} on the image of σ^{r+1} is free. Also, σ is Killing r-rigid if and only if the action of N^{k+r+1} on the image of σ^{r+1} is locally free.*

The following definition of prolongation of Lie subalgebras of $\mathfrak{gl}^{(k)}(n)$ was introduced in [3].

Definition 4. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{gl}^{(k)}(n)$, and let r be a nonnegative integer. The r-th prolongation of $\mathfrak{h}$ is the vector space $\mathfrak{h}_r$ of multilinear transformations $L \in N^{k+r}$ such that $L_{(r)}(v_1, \ldots, v_r) \in \mathfrak{h} \cap N^k$ for every $v_1, \ldots, v_r \in \mathbb{R}^n$.*

Any such Lie subalgebra $\mathfrak{h}$ is said to be of finite type if it has a vanishing prolongation. Moreover, $\mathfrak{h}$ is said to be of finite type of order r, or simply finite type r, if its $(r+1)$-prolongation is the first one that vanishes. Also, if H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$, then H will be said to be of finite type (of order r) if its Lie algebra has the corresponding property.

The next result gives an explicit description of the stabilizers for the actions of N^{k+r} considered above.

Proposition 4. *Let Q be a smooth $\mathrm{Gl}^{(k)}(n)$-space and consider the induced action of $\mathrm{Gl}^{(k+r)}(n)$ on $J^r_n(Q)$, where $r \geq 1$. If $j^r(f) \in J^r_n(Q)$, then the stabilizer of $j^r(f)$ for the action of N^{k+r} is the vector group $\mathfrak{h}_r$, where $\mathfrak{h}$ is the Lie algebra of the stabilizer of $f(0)$ for the action of $\mathrm{Gl}^{(k)}(n)$.*

Proof. Let $L \in N^{k+r}$ be given. It follows from Lemma 1 and its proof that $Lj^r(f) = j^r(\bar{L}f)$, where $\bar{L}\colon \mathbb{R}^n \to N^k$ is given by $\bar{L}(x) = \dfrac{1}{r!}L_{(r)}(x,\dots,x)$. Denote $g = \bar{L}f$ and observe that $j^{r-1}(\bar{L}f) = j^{r-1}(f)$ because $j^{r-1}(\bar{L}) = 0$. On the other hand, given $j^{r-1}(\bar{L}) = 0$, a straightforward computation in local coordinates for Q around $f(0)$ shows that

$$\frac{\partial^r g^i}{\partial x^{j_1}\dots\partial x^{j_r}}(0) = \frac{\partial^r f^i}{\partial x^{j_1}\dots\partial x^{j_r}}(0) + \sum_l \frac{\partial A^i}{\partial y^l}(0, f(0))\frac{\partial^r \bar{L}^l}{\partial x^{j_1}\dots\partial x^{j_r}}(0)$$

for every $i, j_1, \dots, j_r$, where $A\colon \mathrm{Gl}^{(k)}(n) \times Q \to Q$ is the action map, and x^j, y^l denote the natural coordinates on $\mathbb{R}^n$ and N^k, respectively.

It follows from these remarks that the identity $Lj^r(f) = j^r(f)$ is equivalent to the identities

$$\sum_l \frac{\partial A^i}{\partial y^l}(0, f(0))\frac{\partial^r \bar{L}^l}{\partial x^{j_1}\dots\partial x^{j_r}}(0) = 0$$

for every $i, j_1, \dots, j_r$. It follows from the expression for $\bar{L}$ and the fact that A is the action map that each of the sums above is an entry of the matrix representation of the linear map $dR(f(0))_e \circ L_{(r)}\colon \mathbb{R}^n \to T_{f(0)}Q$, where $R(f(0))\colon N^k \to Q$ is given by $h \mapsto hf(0)$. Hence L stabilizes $j^r(f)$ if and only if the image of $L_{(r)}$ is in the kernel of $dR(f(0))_e$. The result then follows because such kernel is the intersection of N^k with the Lie algebra of the stabilizer of $f(0)$ for the action of $\mathrm{Gl}^{(k)}(n)$.

A consequence of these propositions is the following characterization of rigid geometric structures. It shows that the property of being rigid depends only on its type, i.e., it depends only on the $\mathrm{Gl}^{(k)}(n)$-space Q in which the image of the geometric structure lies. It also proves that if a geometric structure is rigid, any sufficiently small perturbation is rigid as well.

Theorem 1. *Let σ be a geometric structure of order k and type Q on an smooth n-dimensional manifold. Then the following conditions are equivalent:*

1. *σ is r-rigid,*
2. *σ is Killing r-rigid,*

3. *there is a* $\mathrm{Gl}^{(k)}(n)$*-invariant open neighborhood* Q' *of the image of* σ *in* Q *so that the stabilizers for the action of* $\mathrm{Gl}^{(k)}(n)$ *on* Q' *are of finite type of order* $\leq r$*. In other words, the Lie algebras of such stabilizers have vanishing* $(r+1)$*-th prolongations.*

Proof. The equivalence of (1) and (2) follows from Proposition 3 and from the fact that, by Proposition 4, the stabilizers of the action of N^{k+r+1} on $J_n^{r+1}(Q)$ are always connected. The equivalence of (2) and (3) follows from Propositions 3, 4 and the fact that the subset Q_0 of $J_n^{r+1}(Q)$ consisting of the points with discrete stabilizer for the N^{k+r+1}-action is an open set whose projection into Q gives the required set Q'. Note that the $\mathrm{Gl}^{(k)}(n)$-invariance of Q' follows from the $\mathrm{Gl}^{(k+r+1)}(n)$-invariance of Q_0, which in turn is a consequence of the normality of N^{k+r+1} in $\mathrm{Gl}^{(k+r+1)}(n)$.

The following result gives a useful improvement of Proposition 3.

Corollary 1. *Let* σ *be a geometric structure of order* k *and type* Q *on an* n*-dimensional smooth manifold. If* σ *is* r*-rigid, then there is a* $\mathrm{Gl}^{(k)}(n)$*-invariant open neighborhood* Q' *of the image of* σ *such that* $J_n^{r+i}(Q')$ *is a* $\mathrm{Gl}^{(k+r+i)}(n)$*-invariant open neighborhood of the image of* σ^{r+i} *on which* N^{k+r+i} *acts freely, for every* $i \geq 1$*.*

Proof. Given the r-rigidity of σ, take Q' as in (3) of Theorem 1. Then Proposition 4 ensures that Q' satisfies the required properties. Note that this uses the fact that $\mathfrak{h}_{r+1} = 0$ implies $\mathfrak{h}_{r+i} = 0$ for every $i \geq 1$.

5. Framed rigidity

The concept of framed rigidity for geometric structures is defined and shown to be equivalent to rigidity. This will require to associate a parallelism to a rigid geometric structure so that the automorphisms and Killing fields of both are closely related.

Definition 5. *A geometric structure* σ *of order* k *on a manifold* M *is framed rigid if for some* $r \geq 0$ *there is a complete parallelism* ω *on* $L^{(k+r)}(M)$ *such that the following two conditions are satisfied:*

1. *if* $\varphi \in \mathrm{Aut}^{\mathrm{loc}}(M, \sigma)$*, then* $\varphi_{(k+r)} \in \mathrm{Aut}^{\mathrm{loc}}(L^{(k+r)}(M), \omega)$*; and*
2. *for every* $s \geq 1$*, if* $j_x^{k+r+s}(\varphi) \in \mathrm{Aut}^{k+r+s}(\sigma, x, y)$ *for some* $x, y \in M$*, then* $j_\alpha^s(\varphi_{(k+r)}) \in \mathrm{Aut}^s(\omega, \alpha, \varphi_{(k+r)}(\alpha))$ *for every* $\alpha \in L^{(k+r)}(M)$ *in the fiber over* x*.*

In such case σ *is also said to be* ω*-framed rigid.*

It is an easy consequence of the definitions that every framed rigid geometric structure is rigid. Theorem 2 will show that the converse is also true.

The relation between automorphisms and Killing fields, both in the local and infinitesimal case, implies that a ω-framed rigid geometric structure σ also satisfies the following properties:

(1′) if $X \in \mathrm{Kill}^{\mathrm{loc}}(M,\sigma)$, then $X_{(k+r)} \in \mathrm{Kill}^{\mathrm{loc}}(L^{(k+r)}(M),\omega)$; and

(2′) for every $s \geq 1$, if $j_x^{k+r+s}(X) \in \mathrm{Kill}^{k+r+s}(\sigma,x)$ for some $x \in M$, then $j_\alpha^s(X_{(k+r)}) \in \mathrm{Kill}^s(\omega,\alpha)$ for every $\alpha \in L^{(k+r)}(M)$ in the fiber over x.

In words, every Killing field of σ, either local or infinitesimal, induces a Killing field for the parallelism ω. As the next result shows, a partial converse holds in the analytic category; thus, to certain extent, the parallelism ω determines the Killing fields of σ.

Lemma 3. *Let σ be an analytic geometric structure on an analytic manifold M. If σ is ω-framed rigid, where ω is an analytic parallelism on $L^{(k+r)}(M)$, then for every $x \in M$ there is an integer $s(x) \geq k+r+1$ such that for every smooth vector field X defined in a neighborhood of x the following are satisfied:*

1. *Let α be an element in the fiber of $L^{(k+r)}(M)$ over x. If $j_x^s(X) \in \mathrm{Kill}^s(\sigma,x)$ and $j_\alpha^{t-k-r}(X_{(k+r)}) \in \mathrm{Kill}^{t-k-r}(\omega,\alpha)$ for some $t \geq s \geq s(x)$, then $j_x^t(X) \in \mathrm{Kill}^t(\sigma,x)$.*
2. *If X, $X_{(k+r)} \in \mathrm{Kill}^{\mathrm{loc}}(L^{(k+r)}(M),\omega)$ and $j_x^s(X) \in \mathrm{Kill}^s(\sigma,x)$, for some $s \geq s(x)$ and $x \in M$, then $X \in \mathrm{Kill}^{\mathrm{loc}}(M,\sigma)$.*

Proof. Since σ is rigid there is some integer s_0 such that the natural projection $\mathrm{Kill}^{t+1}(\sigma,x) \to \mathrm{Kill}^t(\sigma,x)$ is an inclusion for every $t \geq s_0$. This defines, for each $x \in M$, a decreasing sequence of finite dimensional vector spaces, and so there is an integer $s_1(x)$ such that $\mathrm{Kill}^{t+1}(\sigma,x) \to \mathrm{Kill}^t(\sigma,x)$ is an isomorphism for every $t \geq s_1(x)$. Set $s(x) = \max\{s_1(x), k+r+1\}$.

To prove (1), observe that if $t \geq s \geq s(x)$, then there is an infinitesimal Killing field $j_x^t(Y) \in \mathrm{Kill}^t(\sigma,x)$ such that $j_x^s(Y) = j_x^s(X)$. From this it follows that $j_\alpha^{s-k-r}(Y_{(k+r)}) = j_\alpha^{s-k-r}(X_{(k+r)})$ and then the 0-rigidity of the parallelism ω implies that $j_\alpha^{t-k-r}(Y_{(k+r)}) = j_\alpha^{t-k-r}(X_{(k+r)})$. This in turn gives $j_x^t(Y) = j_x^t(X)$, thus showing that $j_x^t(X) \in \mathrm{Kill}^t(\sigma,x)$.

To prove (2), let X be analytic and note that the condition that $X_{(k+r)} \in \mathrm{Kill}^{\mathrm{loc}}(L^{(k+r)}(M),\omega)$ implies that if α is as in (1), then also $j_\alpha^{t-k-r}(X_{(k+r)}) \in \mathrm{Kill}^{t-k-r}(\omega,\alpha)$ for every $t \geq k+r$. Hence, by (1) and the assumption that $j_x^s(X) \in \mathrm{Kill}^s(\sigma,x)$ for some $s \geq s(x)$, it follows that $j_x^t(X) \in \mathrm{Kill}^t(\sigma,x)$ for t sufficiently large. Because both X and σ are analytic this implies that $X \in \mathrm{Kill}^{\mathrm{loc}}(M,\sigma)$.

The next result shows that, for suitable actions, freeness for N^k implies properness for N^{k+1}. It will be required in order to build a parallelism for a given rigid geometric structure.

Lemma 4. *Suppose that* $\mathrm{Gl}^{(k)}(n)$ *acts smoothly on a manifold* Q *and consider the action of* N^{k+1} *on* $J^1_n(Q)$ *induced by the* 1*-prolongation of the* $\mathrm{Gl}^{(k)}(n)$*-action. If* N^k *acts locally freely on* Q*, then* N^{k+1} *acts properly on* $J^1_n(Q)$*.*

Proof. Let $C \subset J^1_n(Q)$ be a compact set and let $(g_i)_i \subset N^{k+1}$ be a sequence such that $g_i C \cap C \neq \phi$ for all i. It has to be shown that $(g_i)_i$ has a subsequence that converges in N^{k+1}.

Choose maps $h_i, \hat{h}_i : \mathbb{R}^n \to Q$ such that $j^1(h_i), j^1(\hat{h}_i) \in C$ and $g_i j^1(h_i) = j^1(\hat{h}_i)$. Assume, after passing to a subsequence if necessary, that $(j^1(h_i))_i$, $(j^1(\hat{h}_i))_i$ converge to $j^1(h)$, $j^1(\hat{h})$, respectively. It follows from Lemma 1 that there are maps $f_i : \mathbb{R}^n \to N^k$ satisfying $f_i(0) = e$ and $g_i j^1(h_i) = j^1(f_i h_i)$. Note that $h_i(0) = \hat{h}_i(0)$ for every i, and so $h(0) = \hat{h}(0)$.

The condition $j^1(f_i h_i) = j^1(\hat{h}_i)$ yields the identity

$$d(h_i)_0 + dR(h_i(0))_e \circ d(f_i)_0 = d(\hat{h}_i)_0$$

for every i, where for $q \in Q$ the map $R(q) : N^k \to Q$ is the orbit map given by $n \mapsto nq$.

The orbit map $R(q)$ is an immersion because the action of N^k on Q is locally free, so $dR(q)_e$ is injective with image $T_q N^k q$ for every $q \in Q$. Hence the above equation shows that $(d(f_i)_0)_i$ converges to $dR(h(0))_e^{-1} \circ (dh_0 - d\hat{h}_0)$ as linear maps $\mathbb{R}^n \to N^k$, where the inverse $dR(h(0))_e^{-1}$ is that of $dR(h(0))_e$ interpreted as a linear map $N^k \to T_{h(0)} N^k h(0)$. It follows that if $f : \mathbb{R}^n \to N^k$ is chosen so that $f(0) = e$ and $df_0 = dR(h(0))_e^{-1} \circ (dh_0 - d\hat{h}_0)$, then $(j^1(f_i))_i$ converges to $j^1(f)$ in $J^1_n(N^k)$. The relation between $(g_i)_i$ and $(j^1(f_i))_i$ and the fact that the inclusion of N^{k+1} into $J^1_n(N^k)$ realizes N^{k+1} as a closed subgroup then imply that $(g_i)_i$ converges in N^{k+1}.

The next result is the key to relate complete parallelisms to rigid geometric structures.

Proposition 5. *Let* $\sigma : L^{(k)}(M) \to Q$ *be a geometric structure of order* k *and type* Q *on* M*. If* σ *is* r*-rigid* ($r \geq 0$)*, then there is a* $\mathrm{Gl}^{(k)}(n)$*-invariant open neighborhood* Q' *of the image of* σ *such that* $J^{r+i}_n(Q')$ *is a* $\mathrm{Gl}^{(k+r+i)}(n)$*-invariant open neighborhood of the image of* σ^{r+i} *on which* N^{k+r+i} *acts freely and properly, for every* $i \geq 2$*.*

Proof. If σ is r-rigid, then Corollary 1 shows that there is a $\mathrm{Gl}^{(k)}(n)$-invariant open neighborhood Q' of the image of σ such that, for $i \geq 1$, the space $J^{r+i}_n(Q')$ is a $\mathrm{Gl}^{(k+r+i)}(n)$-invariant open neighborhood of the image of σ^{r+i} on which N^{k+r+i} acts freely. Lemma 4 applied to the action of $N^{k+r+i-1}$ on $J^{r+i-1}_n(Q')$ implies that the action of N^{k+r+i} on $J^1_n(J^{r+i-1}_n(Q'))$ is proper, for every $i \geq 2$. The remarks at the end of Section 2 provide a $\mathrm{Gl}^{(k+r+i)}(n)$-equivariant embedding of $J^{r+i}_n(Q')$ into $J^1_n(J^{r+i-1}_n(Q'))$ and so we conclude that, for every $i \geq 2$, the space $J^{r+i}_n(Q')$ is a $\mathrm{Gl}^{(k+r+i)}(n)$-invariant open neighborhood of the image of σ^{r+i} on which N^{k+r+i} acts freely and properly.

The main result of this section shows that rigidity implies framed rigidity.

Theorem 2. *Let σ be a smooth geometric structure of order k on M. If σ is r-rigid, then there is a smooth complete parallelism ω on $L^{(k+r+1)}(M)$ such that σ is ω-framed rigid. Moreover, if M and σ are analytic, then ω can be chosen to be analytic.*

Proof. Assume first that σ is smooth. By Proposition 5 there is an open neighborhood Q_1 of the image of σ^{r+2} on which N^{k+r+2} acts freely and properly. If $Q_2 = N^{k+r+2}\backslash Q_1$ denotes the quotient space and π_0 the corresponding quotient map, then Q_2 has a manifold structure such that $\pi_0 \colon Q_1 \to Q_2$ defines a principal N^{k+r+2}-bundle. Because N^{k+r+2} is a vector group, there is a smooth $\{e\}$-reduction Q_0 of the principal bundle $\pi_0 \colon Q_1 \to Q_2$.

The map σ^{r+2} is transversal to Q_0 because Q_0 is a manifold transversal to the N^{k+r+2}-orbits in Q_1 and the image of σ^{r+2} is a union of N^{k+r+2}-orbits. Hence $P = (\sigma^{r+2})^{-1}(Q_0)$ is a smooth submanifold of $L^{(k+r+2)}(M)$.

Let $\pi \colon L^{(k+r+2)}(M) \to L^{(k+r+1)}(M)$ be the natural projection. Because the action of N^{k+r+2} on Q_1 is free, σ^{r+2} maps N^{k+r+2}-orbits diffeomorphically onto N^{k+r+2}-orbits, and this implies that P is transversal to the N^{k+r+2}-orbits in $L^{(k+r+2)}(M)$. It follows from this that the map $\pi|_P \colon P \to L^{(k+r+1)}(M)$ is a diffeomorphism, and so that P is a smooth $\{e\}$-reduction of the principal bundle $\pi \colon L^{(k+r+2)}(M) \to L^{(k+r+1)}(M)$. There is also a natural embedding $L^{(k+r+2)}(M) \subset L(L^{(k+r+1)}(M))$ realizing $L^{(k+r+2)}(M)$ as reduction of the first order frame bundle $L(L^{(k+r+1)}(M))$. It follows that P is an $\{e\}$-reduction of $L(L^{(k+r+1)}(M))$ and so it defines a smooth complete parallelism ω on $L^{(k+r+1)}(M)$.

If φ is a local automorphism of σ, then $\sigma^{r+2} \circ \varphi_{(k+r+2)} = \sigma^{r+2}$. This and $P = (\sigma^{r+2})^{-1}(Q_0)$ imply that $\varphi_{(k+r+2)}(P) = P$. From this it follows easily that $\varphi_{(k+r+1)}$ is a local automorphism of ω.

Choose $s \geq 1$ and let $j_x^{k+r+1+s}(\varphi) \in \mathrm{Aut}^{k+r+1+s}(\sigma, x, y)$. Then $\sigma^{r+2} \circ \varphi_{(k+r+2)} = \sigma^{r+2}$ up to order $s-1$ at every α in the fiber of $L^{(k+r+2)}(M)$ over x. Moreover, in a neighborhood of every point of Q_0 we can write $Q_0 = f^{-1}(0)$ for some vector-valued submersion f, so that locally we also have $P = (f \circ \sigma_{r+2})^{-1}(0)$ and $\varphi_{(k+r+2)}(P) = (f \circ \sigma_{r+2} \circ \varphi^{-1}_{(k+r+2)})^{-1}(0)$. Observe that both $f \circ \sigma_{r+2}$ and $f \circ \sigma_{r+2} \circ \varphi^{-1}_{(k+r+2)}$ are submersions as a consequence of the transversality of σ^{r+2} with Q_0.

Claim. If $f_1, f_2 \colon \mathbb{R}^p \to \mathbb{R}^q$ are submersions with the same $(s-1)$-jet at 0, then for $v = f_1(0) = f_2(0)$ the manifolds $f_1^{-1}(v), f_2^{-1}(v)$ have contact of order $s-1$ at 0.

This claim (which is an easy consequence of the implicit function theorem) and the previous remarks imply that $\varphi_{(k+r+2)}(P)$ and P have contact of order $s-1$ at every α in P that projects down to $y = \varphi(x)$ in M.

Since $(\varphi_{(k+r+1)})_{(1)}|_{L^{(k+r+2)}(M)} = \varphi_{(k+r+2)}$ with respect to the natural inclusion $L^{(k+r+2)}(M) \subset L(L^{(k+r+1)}(M))$, the manifolds $(\varphi_{(k+r+1)})_{(1)}(P)$ and P have contact of order $s-1$ at every point in the fiber over y. Proposition 2 implies that $j^s_\alpha(\varphi_{(k+r+1)}) \in \mathrm{Aut}^s(\omega, \alpha, \varphi_{(k+r+1)}(\alpha))$ for every α in the fiber of $L^{(k+r+1)}(M)$ over x, showing thus that σ is ω-framed rigid.

Finally, observe that if both M and σ are assumed analytic, then the above arguments carry over without change in the analytic category. This is elementary for most of the proof except for the need to trivialize the N^{k+r+2}-bundle $\pi_0 \colon Q_1 \to Q_2$ in the analytic category. However, this is a consequence of Proposition 1.19 in Chapter VIII from [5] which implies, by well known properties of analytic manifolds in the theory of real analytic spaces, that a principal fiber bundle of analytic manifolds is topologically trivial if and only if it is analytically trivial.

6. Killing fields and applications

This section gives applications of the previous results to the study of Killing fields and actions of simple Lie groups. The following is a particular case of Proposition 6.10 in [2].

Proposition 6. *Let M be an analytic manifold and let σ be an analytic, complete parallelism on M. For every $x \in M$ there is an integer $r(x)$ such that if $r \geq r(x)$ and $j^r_x(X) \in \mathrm{Kill}^r(\sigma, x)$, then there is a unique analytic local Killing vector field Y defined in a neighborhood of x so that $j^r_x(Y) = j^r_x(X)$.*

The next result shows that, in the analytic category, infinitesimal Killing fields can be extended to local ones for rigid geometric structures.

Theorem 3. *Let σ be an analytic rigid geometric structure on an analytic manifold M. For every $x \in M$ there is an integer $r(x)$ such that if $r \geq r(x)$ and $j^r_x(X) \in \mathrm{Kill}^r(\sigma, x)$, then there is a unique analytic local Killing vector field $Y \in \mathrm{Kill}^{\mathrm{loc}}(\sigma)$ defined in a neighborhood of x so that $j^r_x(Y) = j^r_x(X)$.*

Proof. Suppose that σ is of order k. By Theorem 2, there is a parallelism ω on $L^{(k+r_0)}(M)$, for some integer r_0, such that σ is ω-framed rigid. Fix $x \in M$ and let $\alpha(x)$ be an element in the fiber of $L^{(k+r_0)}(M)$ over x. Let $r(\alpha(x))$ be the integer given by Proposition 6 applied to the parallelism ω, and, as in the proof of Lemma 3, choose an integer $s_1(x)$ such that the natural projection $\mathrm{Kill}^{t+1}(\sigma, x) \to \mathrm{Kill}^t(\sigma, x)$ is an isomorphism for every $t \geq s_1(x)$. Finally, set $r(x) = \max\{s_1(x), r(\alpha(x)) + k + r_0 + 1\}$.

If $r \geq r(x)$ and $j^r_x(X) \in \mathrm{Kill}^r(\sigma, x)$, then $j^{r-k-r_0}_{\alpha(x)}(X_{(k+r_0)})$ belongs to $\mathrm{Kill}^{r-k-r_0}(\omega, \alpha(x))$, by the ω-framed rigidity of σ. Given this choice of r, Proposition 6 implies that there is a local Killing field Z of ω defined in a neighborhood of $\alpha(x)$ and satisfying $j^{r-k-r_0}_{\alpha(x)}(Z) = j^{r-k-r_0}_{\alpha(x)}(X_{(k+r_0)})$.

Also by the choice of r, for every $l \geq r$ there is a unique infinitesimal Killing field $j^l_x(X^l) \in \mathrm{Kill}^l(\sigma, x)$ such that $j^r_x(X^l) = j^r_x(X)$. Hence both $j^{l-k-r_0}_{\alpha(x)}(Z)$ and $j^{l-k-r_0}_{\alpha(x)}(X^l_{(k+r_0)})$ are in $\mathrm{Kill}^{l-k-r_0}(\omega, \alpha(x))$ and have the same $(r-k-r_0)$-jet. It thus follows from the 0-rigidity of the parallelism ω that $j^{l-k-r_0}_{\alpha(x)}(X^l_{(k+r_0)}) = j^{l-k-r_0}_{\alpha(x)}(Z)$, for l large enough. Note that $X^l_{(k+r_0)}$ is $\mathrm{Gl}^{(k+r_0)}(n)$-invariant and preserves the canonical form on $L^{(k+r_0)}(M)$, and so Z satisfies these same properties up to order l at $\alpha(x)$. Since this holds for every large enough l, and since Z, the canonical form on $L^{(k+r_0)}(M)$ and the $\mathrm{Gl}^{(k+r_0)}(n)$-action are all analytic, it follows that Z is $\mathrm{Gl}^{(k+r_0)}(n)$-invariant and preserves the canonical form on $L^{(k+r_0)}(M)$ in a neighborhood of $\alpha(x)$. It follows from the properties of higher order frame bundles that there exists an analytic vector field Y defined in a neighborhood of x such that $Y_{(k+r_0)} = Z$ in a neighborhood of $\alpha(x)$ (cf. Lemma 4.4 in [2]). The identities $j^{l-k-r_0}_{\alpha(x)}(Y_{(k+r_0)}) = j^{l-k-r_0}_{\alpha(x)}(Z) = j^{l-k-r_0}_{\alpha(x)}(X^l_{(k+r_0)})$ imply $j^l_x(Y) = j^l_x(X^l)$ thus showing that $j^l_x(Y) \in \mathrm{Kill}^l(\sigma, x)$ for l large enough. This, together with $Y_{(k+r_0)} = Z \in \mathrm{Kill}^{\mathrm{loc}}(L^{(k+r_0)}(M), \omega)$ and Lemma 3, shows that $Y \in \mathrm{Kill}^{\mathrm{loc}}(M, \sigma)$. Moreover, since $j^r_x(X) = j^r_x(X^r) = j^r_x(Y)$ we conclude that Y is an extension of $j^r_x(X)$ to a local Killing field of σ.

Finally, uniqueness is easily proved by using the corresponding property for the complete parallelism ω.

The next result allows to extend, in the analytic category, local Killing fields to global Killing fields. It is proved by adapting the arguments in Amores [1] to a parallelism obtained from framed rigidity.

Proposition 7. *Let σ be an analytic rigid geometric structure on an analytic simply connected manifold M. Then every local analytic Killing field of σ defined in a connected open subset of M extends to a unique global analytic Killing field.*

As an application, we obtain Gromov's centralizer theorem for arbitrary rigid unimodular geometric structures of algebraic type on finite volume analytic manifolds. We refer to [2] for the definition of a Zariski measure; here we simply observe that any smooth measure on an analytic manifold is a Zariski measure. Also, recall that a geometric structure of type Q and order k is said to be of algebraic type if both Q and the $\mathrm{Gl}^{(k)}(n)$-action on Q are real algebraic.

Theorem 4. *Let M be a connected analytic manifold endowed with an analytic rigid geometric structure σ of algebraic type.*

Let G be a connected, noncompact, simple Lie group acting analytically on M, preserving both σ and a finite Zariski measure.

Let $\mathcal{G}$ be the Lie algebra of Killing vector fields on the universal cover $\widetilde{M}$ induced by the action of the universal cover $\widetilde{G}$ of G. If $\mathcal{V}$ denotes the space of analytic Killing vector fields on $\widetilde{M}$ that centralize $\mathcal{G}$, then

1. *$\mathcal{V}$ is $\pi_1(M)$-invariant,*
2. *$\mathcal{V}$ is finite dimensional,*
3. *there exists an open, conull subset $\widetilde{U}$ of $\widetilde{M}$, invariant under both $\widetilde{G}$ and $\pi_1(M)$, on which $\widetilde{G}$ acts locally freely and such that $\mathrm{ev}_x(\mathcal{V}) \supset T_x\widetilde{G}x$ for every $x \in \widetilde{U}$.*

Here, for a point $x \in M$ and a vector field X defined in a neighborhood of x, $\mathrm{ev}_x(X) = X_x$ is the evaluation map. Also, if the measure on M is smooth, then $\widetilde{U}$ can be assumed to be dense.

Proof. This is obtained following the arguments in [2, Section 9] and using Theorem 3 and Proposition 7.

This theorem has a number of consequences relevant for the study of actions of simple Lie groups. Two of the main corollaries are stated below and further results can be found in [2, Section 10], all of whose statements extend to the present setting of arbitrary analytic rigid unimodular geometric structures of algebraic type on finite volume manifolds. Their proofs follow the same arguments discussed in [2, Section 10] with the use of Theorem 4. More details on the applications of Gromov's centralizer theorem can be found in Zimmer [10,11] and Spatzier-Zimmer [9].

In the following statements, the manifold M and the group G are as in Theorem 4 and are assumed to satisfy its hypothesis.

Theorem 5. *There is a representation $\rho\colon \pi_1(M) \to \mathrm{Gl}(q)$ such that the Zariski closure of $\rho(\pi_1(M))$ contains a subgroup locally isomorphic to G.*

Theorem 6. *Suppose that G has finite center and finite fundamental group. Then the action of G on M is topologically engaging on a conull, open subset of M. Moreover, there exists a conull, open set $\widetilde{U} \subset \widetilde{M}$, which is invariant under both $\widetilde{G}$ and $\pi_1(M)$, and such that the $\widetilde{G}$-orbit of each of its points is closed in $\widetilde{U}$. Furthermore, if the measure is smooth, then the open sets in both M and $\widetilde{M}$ where the topological engagement condition is satisfied can be assumed to be dense.*

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