Foliations and the Higson Compactification

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joint work with J. A. Alvarez Lopez

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Compactifications

- A compactification of a topological space, X, is a pair (X^κ, κ) consisting of a Hausdorff topological space X^κ and an embedding $\kappa : X \to X^\kappa$ with open, dense image.
- Thus if X admits a compactification, then it is locally compact.
- The complement of the image $\kappa(X)$ in X^κ is called the **corona** or **growth** of the compactification, and denoted by κX.

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Examples

- One point or Alexandroff compactification X^{∞}. The corona is a single point (if X is noncompact)
- Stone-Čech compactification X^{β}. The corona is a very complex space.
- Endpoint or Freudenthal compactification X^{end}. The corona is a totally disconnected space.

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Methods of constructing compactifications

- One method of constructing compactifications of a space X is via Banach subalgebras of $C_b(X)$, the Banach algebra of bounded continuous functions on X. To any such algebra, $\mathcal{A}$, associate the evaluation mapping $e_{\mathcal{A}} : X \to \prod_{f \in \mathcal{A}} [\inf f, \sup f]$. If $\mathcal{A}$ contains the constant functions and generates the topology of X, then $(\overline{e_{\mathcal{A}}(X)}, e_{\mathcal{A}})$ is a compactification of X (Čech).
- Another version is via the maximal ideal space of $\mathcal{A}$ (Stone).
- Another method uses ultrafilters of sets in a given ring of closed subsets of X (Wallman-Frink compactifications).
- There are some old open problems on the relationship between the two methods.

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Examples of algebras and compactifications

- The Stone-Čech compactification corresponds to the algebra $C_b(X)$.
- The one-point compactification corresponds to the algebra generated by the functions that are constant on the complement of a compact subset of X.
- The endpoint compactification corresponds to the algebra generated by the (bounded, continuous) functions that are locally constant on the complement of a compact subset of X.
- If X is a dense leaf of compact foliated space F, the algebra generated by the continuous functions on F and the compactly supported functions on F corresponds to a compactification of X that is a foliated space and has F as corona.

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The Higson compactification

Consider a metric space (X, d).

- For a function f on X and a real number $r > 0$, let $\nabla_r f(x) = \sup\{|f(x) - f(y) \mid d(x, y) \leq r\}$
- The Higson algebra of (X, d), denoted by $C_\nu(X)$, is the subalgebra of $C_b(X)$ generated by the functions f such that, for each $r > 0$, $\nabla_r f(x) \to 0$ as $x \to \infty$ on X.
- The Higson compactification of X is denoted by X^ν. The Higson algebra contains the algebra that determines the endpoint compactification of X, so the there is a continuous mapping $X^\nu \to X^{\mathrm{end}}$ that is the identity on X.
- The Higson algebra and compactification was introduced by Higson in his work on index theorems. It was further studied by Higson and Roe (Analytic K-homology). Hurder has studied it in the context of index theorems for foliated spaces.

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Structure of the Higson compactification

Let (X, d) be a non compact proper metric space.

- X^{ν} is much larger than X^{end}. In fact, (for a non compact proper metric space X), no point of the Higson corona is a G_{δ}-set.
- Let x_n be a sequence of points in X diverging to ∞ and $r_n > 0$ a sequence of real numbers such that the metric balls $B(x_n, r_n)$ are mutually disjoint. Then the function

$$f(x) = \begin{cases} \dfrac{r_n - d(x, x_n)}{r_n} & \text{if } d(x, x_n) < r_n \\ 0 & \text{otherwise} \end{cases}$$

  is a Higson function on X.
- If $U \subset X^{\nu}$ is a neighborhood of a point p in the Higson corona νX, then $U \cap X$ contains metric balls of arbitrarily large radius. Conversely, if $W \subset X$ contains metric balls of arbitrarily large radius, then the closure of W in X^{ν} is a neighborhood of some point p in νX.

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Limit Sets

- Let X be a foliated space, let F be a leaf of X, and let F^γ be a compactification of F with corona $\gamma F = F^\gamma \setminus F$.
- The **limit set** of a point e in the corona γF, denoted by $\lim(e)$, is the cluster set of the inclusion mapping $F \hookrightarrow X$ at e, that is
$$\lim(e) = \bigcap_{U \in \mathcal{U}_e} \mathrm{Cl}_X(U \cap F),$$
where $\mathcal{U}_e$ denotes the collection of open neighborhoods of e in F^γ.
- The limit set $\lim(e)$ is a closed subset of X, which may or may not be saturated (union of leaves).
- Let X be a foliated space whose leaves are endowed with a continuous complete metric (e.g. X is compact). If F is a leaf of X and e is a point in νF, then $\lim(e)$ is a saturated subset of X.

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Higson recurrence

A leaf, F, of a foliated space, X, is **Higson recurrent** if the limit point of each point in the Higson corona of F is X.

Theorem

Let X be a compact foliated space. The following are equivalent:

1. *X is minimal (every leaf is dense).*
2. *There is a Higson recurrent leaf.*
3. *Every leaf is Higson recurrent.*

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Generic topology. End homogeneity

Ghys and later Cantwell and Conlon studied the generic topology of leaves of foliated spaces.

Theorem (Ghys)

Let X be a compact foliated space, μ an ergodic harmonic measure. Then there is a saturated set of full measure $Y \subset X$ such that:

1. *every leaf in Y is compact, or*
2. *every leaf in Y has one end, or*
3. *every leaf in Y has two ends, or*
4. *every leaf in Y has a Cantor set of ends*

Theorem (Cantwell-Conlon)

Let X be a compact foliated space with an end recurrent leaf. Then there is a residual saturated set $Y \subset X$ satisfying one of the properties (1), (2), (3) or (4) in Ghys' theorem.

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Higson recurrence and homogeneity

- A topological space, X, is **weakly homogeneous** if for all x, y in X, every neighborhood of x contains an open subset homeomorphic to a neighborhood of y.
- Two spaces, X and Y, are weakly homogeneous if the disjoint union $X \sqcup Y$ is weakly homogeneous.
- Two weakly homogeneous spaces have the same topological dimension.

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Theorem

Let X be a minimal, compact foliated space. Then the Higson corona of any two leaves without holonomy are weakly homogeneous.

- By Hector, and Epstein, Millet, and Tischler, the set of leaves without holonomy is a residual saturated set
- The proof of the theorem uses:
  - the general topological structure
  - the Higson recurrence of the leaves of a minimal foliated space,
  - the local stability of a foliated space that allows to lift, with small distortion, large pieces of leaves to nearby leaves.

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Algebraic Characterization of Spaces

Theorem (Gelfand)

Two locally compact Hausdorff spaces, X and Y, are homeomorphic if and only if the Banach algebras $C_b(X)$ and $C_b(Y)$ are isomorphic.
In fact, an algebraic isomorphism $C_b(Y) \to C_b(X)$ induces a homeomorphism $X^\beta \to Y^\beta$ that sends X to Y.

Algebraic characterization of geometric structures

- Let R be a Riemann surface. The Royden algebra of R, denoted by $M(R)$, is the algebra generated by by the continuous functions f on R that have finite Dirichlet integral $D(f) < \infty$, where

$$D(f) = \int_R df \wedge \star df$$

Theorem (Nakai)

Two Riemann surfaces, R and R', are quasi-conformally equivalent if and only if the Royden algebras $M(R)$ and $M(R')$ are algebraically isomorphic.

- Nakai and Lelong-Ferrand have extended this theorem to Riemannian manifolds and Lewis to domains in euclidean space.
- Recently, Bourdon has given an algebraic characterization of the property of two metric spaces being homeomorphic via a quasi-Mobius homeomorphism.

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Algebraic Characterization of coarsely quasi-isometric spaces

- Two metric spaces, X and X', are coarsely quasi-isometric if there is a bi-Lipschitz bijection between some nets $A \subset X$ and $A' \subset X'$.
- $\mathbf{R}$ and $\mathbf{Z} \subset \mathbf{R}$ are coarsely quasi-isometric.
- A map $f : X \to X'$ is large scale bi-Lipschitz if there are constants $\lambda \geq 1$ and $c > 0$ such that

$$(1/\lambda)d(x,y) - c \leq d'(f(x), f(y)) \leq \lambda d(x,y) + c$$

- $f : X \to X'$ is a large scale bi-Lipschitz equivalence if there is a large scale bi-Lipschitz map $g : X' \to X$ such that $\sup_{x \in X} d(gf(x), x) < \infty$ and $\sup_{x' \in X'} d'(fg(x'), x') < \infty$.

These concepts were introduced by Gromov and developed by many others.

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These concepts were introduced by Gromov and developed by many others.

Algebraic Characterization of coarsely quasi-isometric spaces

- Two metric spaces, X and X', are coarsely quasi-isometric if there is a bi-Lipschitz bijection between some nets $A \subset X$ and $A' \subset X'$.
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Let (X, d) and (X', d') be proper, length metric spaces. An algebraic isomorphism $C_\nu(X') \to C_\nu(X)$ induces a large scale bi-Lipschitz equivalence $X \to X'$.

- As stated, the theorem is not very satisfactory because the conclusion is stronger than desired.
- Indeed, the algebraic isomorphism of Higson algebras induces induces a homeomorphism of Higson compactifications that sends $X \to X'$ (because the non-G_δ property of points in the Higson corona).
- The resulting map $X \to X'$ is in fact a homeomorphism. Coarse quasi-isometries $X \to X'$ are not necessarily defined on the whole space.
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Coarse structures

These are structures that have been introduced by Roe and further studied by Higson and Roe, Block-Weinberger, Hurder.

- A coarse structure on a set X is a correspondence that assigns to any set S an equivalence relation (being close) on the set of mappings $S \to X$ such that
  1. if $p, q : S \to X$ are close and $h : S' \to S$ is any map, then $p \circ h$ and $q \circ h$ are close
  2. Finite unions of close maps are close.
  3. Any two constant maps $S \to X$ are close.
- A subset $E \subset X \times X$ is controlled if the projections $p_1, p_2 : E \to X$ are closed. A subset B is bounded if $B \times B$ is controlled.
- A metric space, (X, d), has a natural coarse structure given by declaring two maps $p, q : S \to X$ to be close if $\sup_{s \in S} d(p(s), q(s)) < \infty$.

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The coarse category

The following concepts can be introduced in the general context of coarse space. Let (X, d) and (X', d') be metric spaces.

- A map $f : X \to X'$ is a coarse map if it satisfies the following:
  1. Uniformly expansive: for each $R > 0$ there is an $S > 0$ such that if $f(x, z) \leq R$, then $d'(f(x), f(z)) \leq S$.
  2. Metric properness: if $B \subset X'$ is bounded, then $f^{-1}B \subset X$ is bounded.
- X and X' are coarsely equivalent if there are coarse maps $f : X \to X'$ and $g : X' \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity mappings 1_X and $1_{X'}$ respectively.
- Being large scale bi-Lipschitz equivalent is weaker than being coarsely quasi-isometric.
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Higson algebra of bounded functions

- Let X be a coarse proper metric space. A bounded function $f : X \to \mathbf{R}$ is a Higson function if, for each $r > 0$, $\nabla_r f(x) \to 0$ as $x \to \infty$ in X. Let $\mathcal{B}_\nu(X)$ be the set of Higson functions on X endowed with the supremum norm.
- The Higson compactification X^ν of X was constructed as the maximal ideal space of the Higson algebra of continuous functions $C_\nu(X)$.
- It turns out that X^ν is also the maximal ideal space of the Banach algebra $\mathcal{B}_\nu(X)$ because any $f \in \mathcal{B}_\nu(X)$ has an extension to X^ν that is continuous on the points of the corona νX.
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Boundary extension of coarse mappings

Theorem

Let X and X' be proper metric spaces.

1. *A map $f : X \to X'$ is coarse if and only if it has an extension $f^\nu : X^\nu \to X'^\nu$ that is continuous on the points of νX and sends νX into $\nu X'$.*
2. *Two coarse maps $f, g : X \to X'$ are close if and only if the extensions f^ν and g^ν given in (1) are equal on νX.*
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