Dynamics on Spaces of Higher Order Random Structures

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[joint work with
Jesús Álvarez López (USC)]

Random Shapes - IPAM -

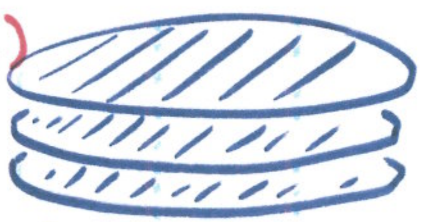
Lake Arrowhead - Dec 7-12, 2008

Dynamical System

Geometry of Orbits

Locally

$\frac{dy}{dx} = f(x,y)$



local orbits

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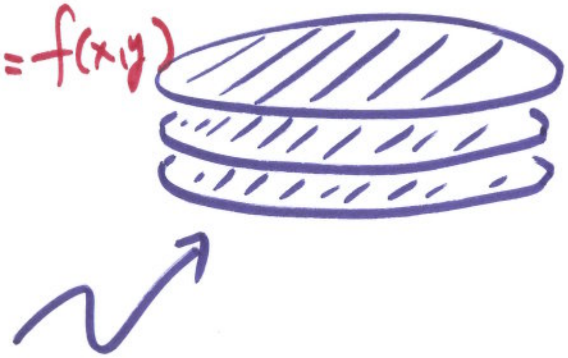

local orbits

Globally

Question I:

When do all orbits look alike?

(Technical concept
" Quasi-isometric"
" Finite G.H distance")

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Theorem (Alvarez Lopez - C) On a transitive dyn. system

(a) Either all orbits "look alike"

or

(b) There are uncountable many different looking orbits

__Theorem__ (AL-C) On a transitive dyn. system (almost) all orbits look alike $\iff$ there is an orbit which is "highly homogeneous"

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"Except for those trajectories that are obviously simple, almost all of the trajectories of a dynamical system can be viewed as computations of equivalent sophistication, which is to say that they are universal"

S. WOLFRAM, A New Kind of Science Ch. 12

Question II

What geometric properties are common to all orbits?

QI - invariants

GH - invariants

Examples:

→ Volume growth (Hector, Alvarez Lopez - C)

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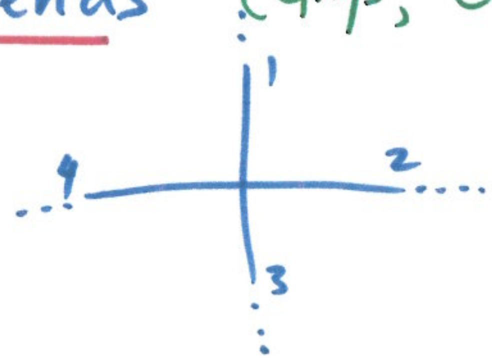
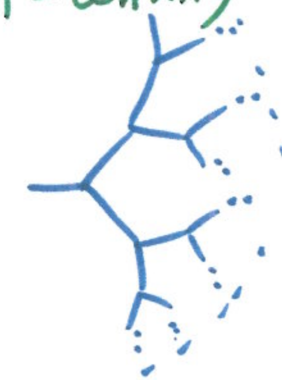
→ Number of ends (Ghys, Cantwell - Conlon)





Cantor set

Examples:

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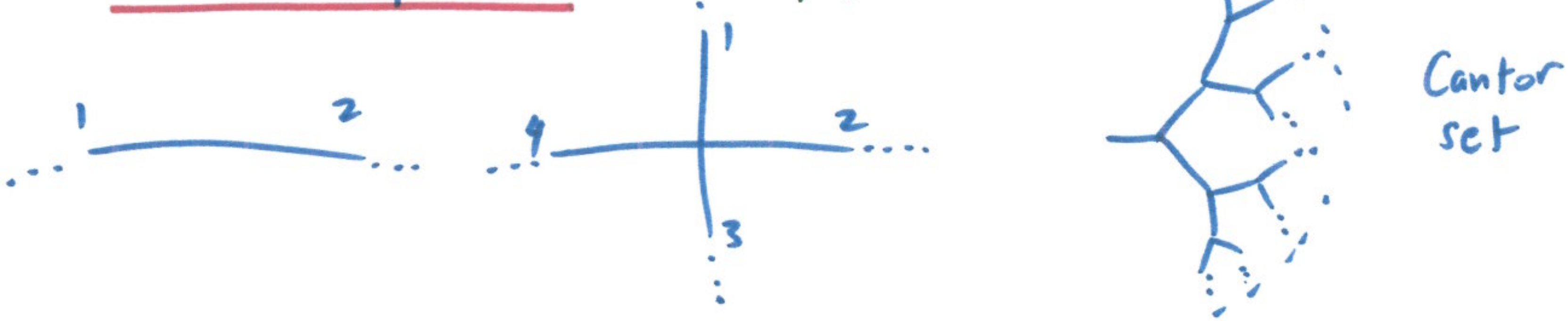
→ Number of ends (Ghys, Cantwell - Conlon)



→ Spectrum of Laplacian (Hurder)

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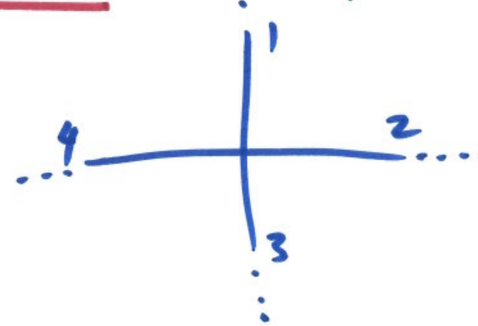
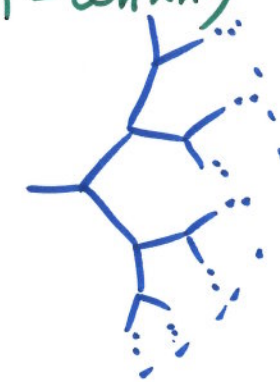




Cantor set

→ Spectrum of Laplacian (Hurder)

→ Asymptotic dimension (AL - C)

→ Homology at ∞ (AL - C)

Theorem (Alvarez Lopez - C)

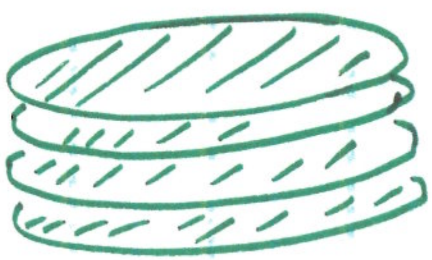
If X is a (transitive) dynamical system then (almost) all its orbits share the same geometric invariants.

(must define)

General Question

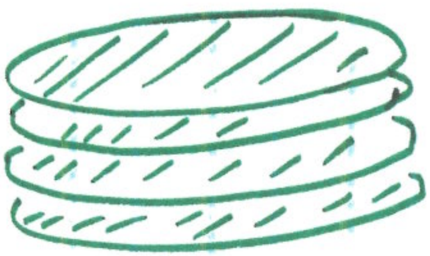
How does

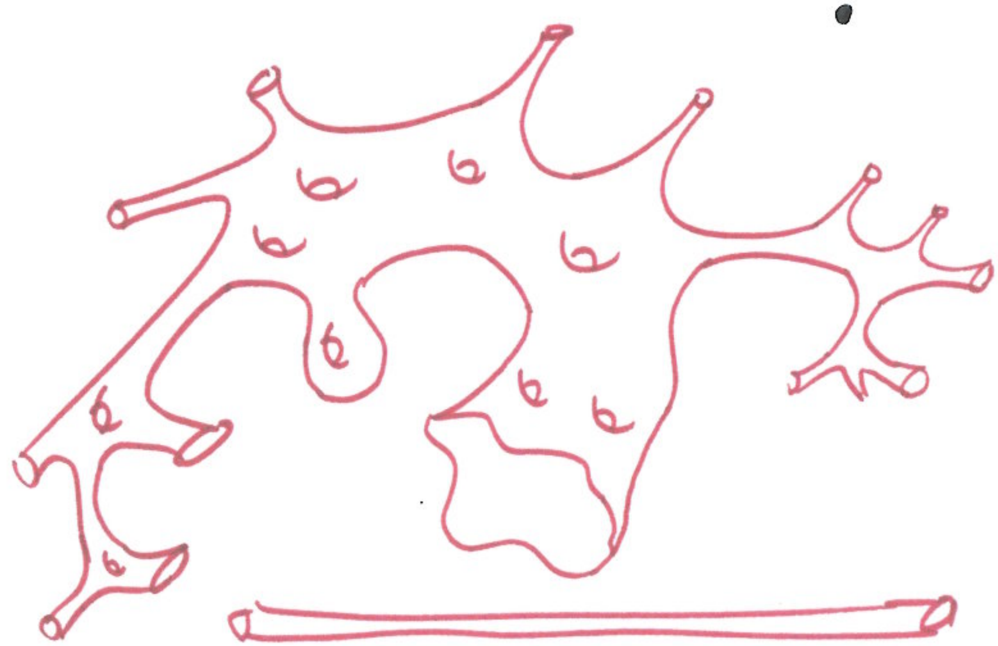
"local sameness"



General Question

How does "local sameness" CONTROL "distant random behavior"?





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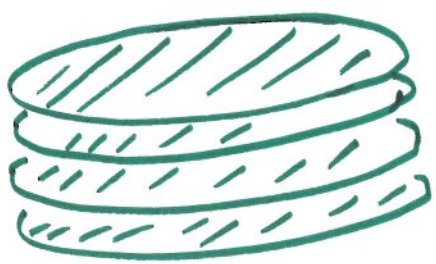

Answer: $\mathcal{M}_*$ space of Higher order Random Structures

Hausdorff distance between compact subsets

X, Y of a metric space

$$\mathrm{dist}_H(X,Y) = \inf\left\{ \varepsilon > 0 \;\middle|\; \begin{array}{l} X \subset B_\varepsilon(Y) \\ Y \subset B_\varepsilon(X) \end{array} \right\}$$

Hausdorff distance between compact subsets X, Y of a metric space

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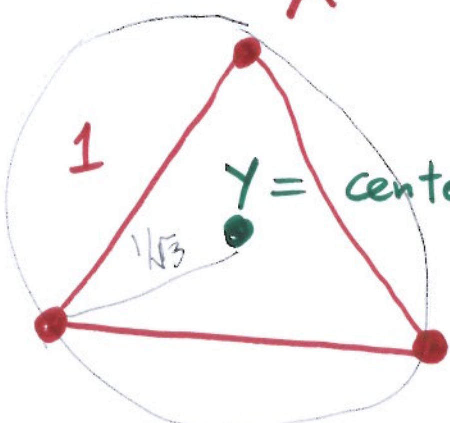
Example:

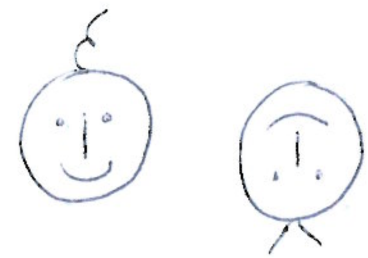
$X =$ unit equilateral triangle in $\mathbb{R}^2$



$Y =$ center of mass

$$\mathrm{dist}_H(X,Y) = \frac{1}{\sqrt{3}}$$

Gromov-Hausdorff distance

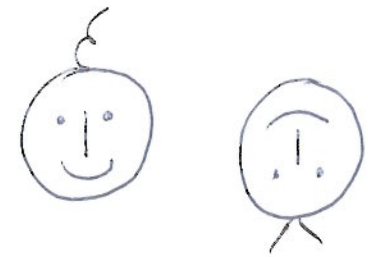


For compact metric spaces M, N

$$\mathrm{dist}_{GH}(M,N) = \inf \{ \text{Hausdorff } \mathrm{dist}_H(M,N) \}$$

where "inf" is taken over all metrics on disjoint union $M \sqcup N$ inducing given metrics on M, N

Gromov-Hausdorff distance



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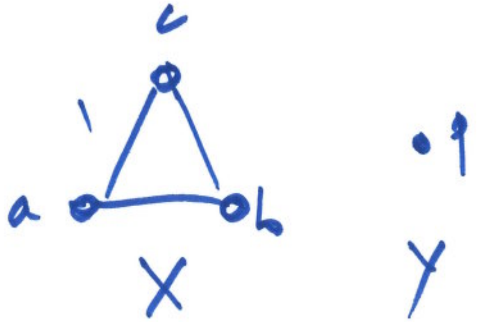
Example:

$$\mathrm{dist}_{GH}\left(\underset{X}{\triangle}, \underset{Y}{\bullet}\right) = \frac{1}{2}$$



$d_H \geq \frac{1}{\sqrt{3}}$

Find $\underline{\underline{d}}$ on $X \sqcup Y$

$d(a,p) = 1/2$

$d(b,p) = 1/2$

$d(c,p) = 1/2$

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but dist_{GH} does not behave well on non-compact spaces (like orbits of dyn. system)

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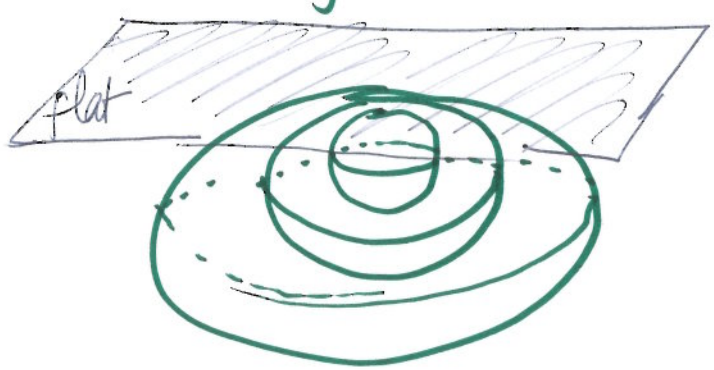
but $dist_{GH}$ does not behave well on non-compact spaces (like orbits of dyn. system)

Example:

Spheres of larger and larger radius "look like" flat space

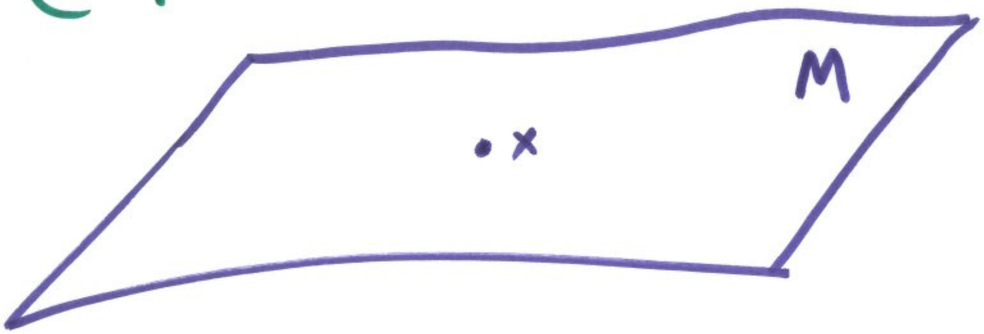


but do not approach it with $dist_{GH}$

$$dist_{GH}(\text{sphere}, \mathbb{R}^n) = \infty$$

$\mathcal{M}_*$ = isometry classes of complete proper, pointed metric spaces with Gromov-Hausdorff topology

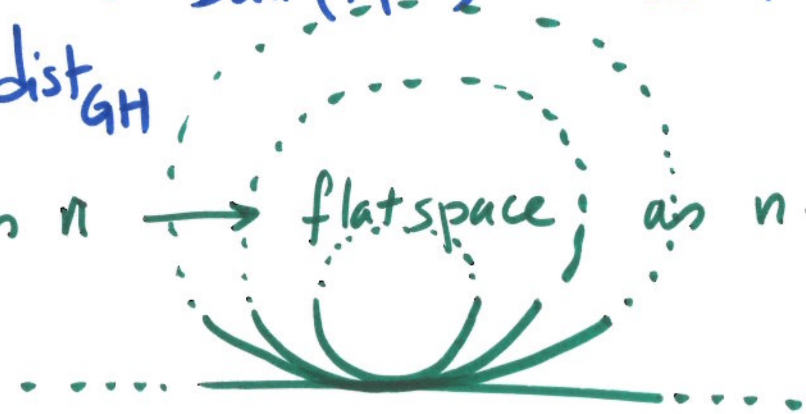
(locally compact)



$[M, x]$

$[M_k, x_k] \longrightarrow [M, x]$ if for every radius $r > 0$

$$\mathrm{Ball}(x_k, r) \xrightarrow[\mathrm{dist}_{GH}]{} \mathrm{Ball}(x, r) \quad \text{as } k \to \infty$$



Example: Sphere radius $n \longrightarrow$ flatspace as $n \to \infty$

$\mathcal{M}_*$ is metrizable, separable, complete

Gromov-Hausdorff dist_{GH} is not a distance on $\mathcal{M}_*$

but defines an equivalence relation

$$(M,x) \underset{GH}{\sim} (N,z) \iff \mathrm{dist}_{GH}\big((M,x),(N,z)\big) < \infty$$

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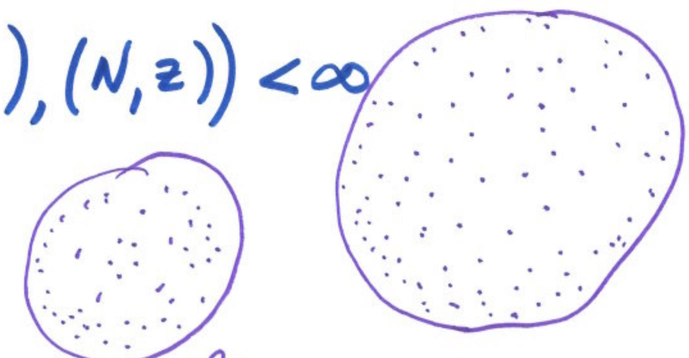
$$(M,x) \sim_{GH} (N,z) \iff dist_{GH}((M,x),(N,z)) < \infty$$

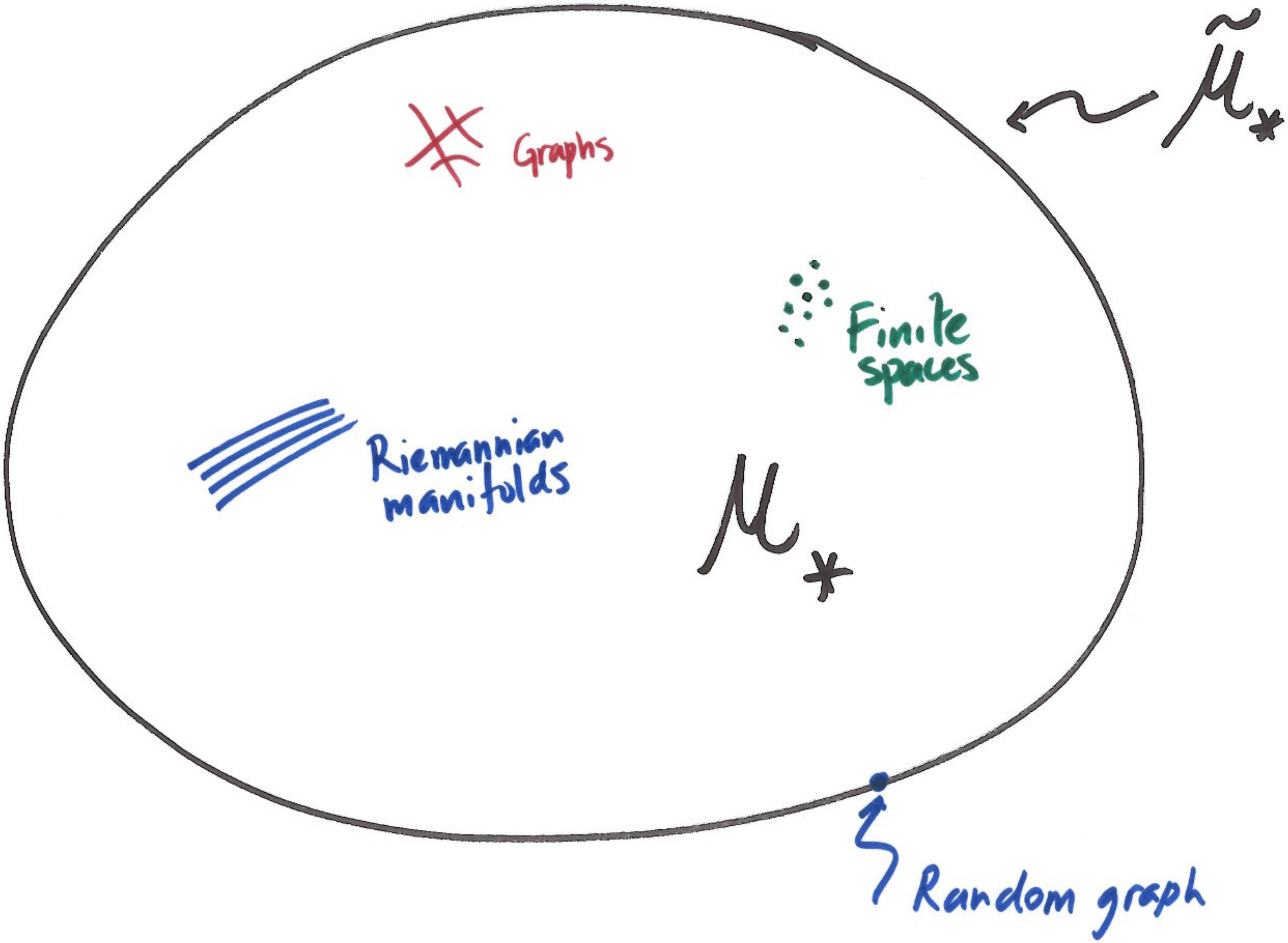
There is a "coarse version" $dist_{QI}$ (quasi-isometry distance)

$$(M,x) \sim_{QI} (N,z) \iff dist_{QI}((M,x),(N,z)) < \infty$$

$$\boxed{\sim_{GH} \Rightarrow \sim_{QI}}$$



$\frac{1}{2}\mathbb{H}^2 \leftsquigarrow \mathbb{H}^2$



Graphs
Finite spaces
Riemannian manifolds
$\mathcal{M}_*$
$\tilde{\mathcal{M}}_*$
Random graph

Geometric invariants of spaces are

$$\Phi : \mathcal{M}_* \longrightarrow Y$$

which are constant on

QI equiv. classes

(or)

GH equiv. classes

$\Big($ Φ nice map, Y nice space $\Big)$

measurable Borel, separable

X dynamical system gives rise to map

$$i : X \longrightarrow \mathcal{M}_*$$

$$x \longmapsto (\text{Orbit}\, x, x)$$

which is Borel (measurable) (sometimes continuous)

Reason: "local sameness"

Setting: X dynamical system

$\Phi : \mathcal{M}_* \longrightarrow Y$ geometric invariant

composition

$$X \xrightarrow{i} \mathcal{M}_* \xrightarrow{\Phi} Y$$

is constant on orbits of X

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$\Phi : \mathcal{M}_* \to Y$ geometric invariant

Composition $X \xrightarrow{i} \mathcal{M}_* \xrightarrow{\Phi} Y$

is constant on orbits of X

Consequence: X transitive (orbits tightly wrapped)

Φ Borel, Y separable

then Φ is the same for (almost) all orbits of X

Gromov-Hausdorff space $\mathcal{M}_*$

(space of higher order random structures)

is a dynamical system (in several natural ways)

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dilation flow $dist \mapsto \lambda\, dist$

$\boxed{\text{How complex is } \mathcal{M}_* \text{ as dynamical system?}}$

Equivalence relation classification

Compare equiv. relations

simple (id) — complex (irrational rotation) — more complex ($\mathcal{U}_*$) (tail $\sim$)

Group action:

Are $\sim_{GH}$, $\sim_{QI}$ comparable to orbit equiv. relation of group action?

Classifiable by countable structures

$(X, \sim)$ $\quad x \longmapsto S_x$ countable set

$$x \sim y \iff S_x \equiv S_y$$

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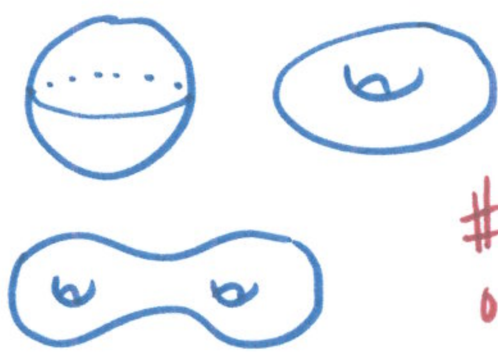
Examples:

Surfaces (homeo)



$\#$ handles

orientation

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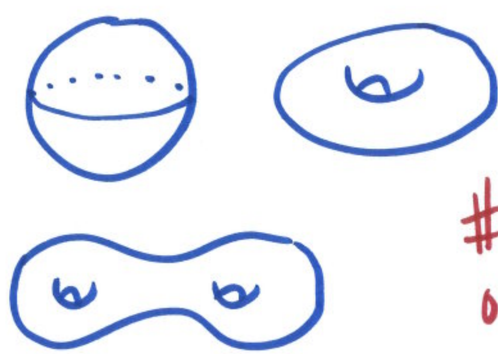

handles
orientation

Inv. matrices (conjugacy)

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

eigenvalues

Theorem (Kechris)

Orbit equiv. relation of action of locally compact group is classifiable by countable structures

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Theorem (Hjorth)

For orbit equiv. relation of group action TFAE

(1) Classifiable by countable structures

(2) Action is not turbulent

Turbulence

(Hjorth $\to$ group actions)

(Alvarez Lopez - C $\to$ metric equiv. relations)

$X, \sim$ metric equiv. relation is turbulent

(1) Equivalence classes (orbits) are dense

(2) Equivalence classes are meager

(3) Local equiv. classes are locally dense

Turbulence (Hjorth → group actions)

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Example: $\mathbb{R}^\infty \equiv$ sequences of real numbers

$(x_n) \sim (y_n) \iff \exists N$ such that (some tail)

$x_n = y_n$ for $n \geq N$

Theorem (Alvarez López - C)

$\mathcal{M}_*$ Gromov-Hausdorff space

$\sim_{GH} \equiv$ finite Gromov-Hausdorff distance

(1) $\sim_{GH}$ is turbulent

(2) $\sim_{GH}$ not classifiable by countable structures

(3) $\sim_{GH}$ not reducible to group action

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Conjecture Same is true for $\sim_{QI}$

finite quasi-isometric distance