

# **Foliations II**

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To our wives Juana and Jackie

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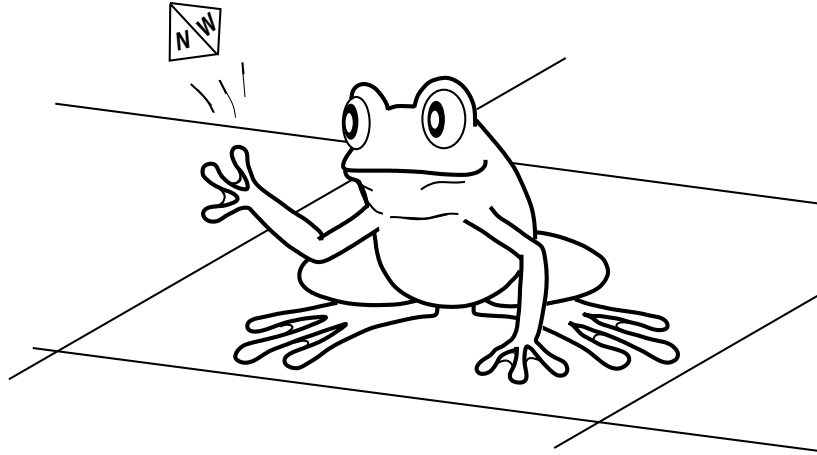


Figure C.7.1. Random frog

If the behavior of the frog is accepted as a discrete version of the movement of a Brownian particle, then it is reasonable to expect that the solution to the Dirichlet problem on a bounded domain  $D$  of the manifold  $X$  with boundary data  $\varphi$  will be given by

$$f(x) = E_x [\varphi(\omega(T_D(\omega)))]$$

where  $T_D$  is the first exit time from  $D$ .

The random frog will now be put to work toward a solution to the Poisson problem, submitting her to the following process. Positioned at time 0 at the point  $(mq, nq)$ , let her jump at will (at discrete times  $t = 0, 1, 2, \dots$ ) to one of the neighbouring lily pads with the same probability as before. If at time  $T$  she hits a boundary pad, then assign the first exit time  $T = T(\omega)$  to the sample Brownian path. While it may or may not be possible to explicitly compute the expectation  $E_{(m,n)}[T]$ , it turns out that it satisfies an important identity.

As before, if the frog is at  $(mq, nq)$  at time  $t$ , then at time  $t + 1$  she is going to be at one of the neighboring lilies  $(m'q, n'q) = ((m + s)q, (n + s)q)$  with probability  $1/4$ . It follows that

$$E_{(m,n)}[T] = \left( \sum_{-1 \leq r, s \leq 1} p_{rs} E_{(m+r, m+s)}[T] \right) + 1.$$

Equivalently, the function  $f(mq, nq) = E_{(m,n)}[T]$  satisfies the equation

$$\Delta f(mq, nq) = -1$$