Some examples of exit times

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1 Introduction

The theory of Brownian motion allows probabilistic interpretatiosn of solutions to second order differential equations. One such example is the folloging. If D is a regular domain on a manifold M with laplace operator Δ then a solution to the differential equation

$$\Delta f = -1$$

on D with boundary condition $f \equiv 0$ on ∂D has the probabilistic interpretation $f(x) = E_x[T_D]$, where E_x is the expected value functional on the Wiener space of paths ω in M with $\omega(0) = x$, and T_D is the first exit time from D, mamely $T_D(\omega) = \inf\{t > 0 \mid \omega(t) \notin D\}$ (with the standard convention that the infimum of the empty set is ∞). More details on exit times and differential equations can be found in Dynkin [2]; a brief survey in [1, Appendix C].

In this note we study some examples of exits times from domains in the Euclidean and hyperbolic planes. Ghys [3] gave a spectacular aplication of exit times in geometry. The examples here are the base of some exercises in [1, Exercises 2.7.14 C.9.4 and C.9.5], and the calculations were motivated by a discussion on Ghys theorem.

2 Cones in the Euclidean plane

In cartesian coordinates (x, y) the euclidean laplacian has the expression

$$\Delta f = f_{xx} + f_{yy}$$

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and in polar coordinates (r, θ)

$$\Delta f = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}.$$

Let C be a cone of opening angle 2α in the euclidean plane. In polar coordinates, C is, up to isometry, the set of those points (r, θ) with $r > 0$ and $\theta \in [-\alpha, \alpha]$.

A solution f to the equation $\triangle f = -1$ in C satisfies

$$f(\lambda x, \lambda y) = \lambda^2 f(x, y),$$

and in polar coordinates:

$$f(\lambda r, \theta) = \lambda^2 f(r, \theta).$$

This says that f is completely determined by the values $f(1, \theta)$, and it can be written as $f(r, \theta) = r^2 h(\theta)$, where h is positive and symmetric on $[-\alpha, \alpha]$. Moerover, h satisfies the differential equation

$$h'' + 4h = -1$$

or in a different way:

$$(h + 1/4)'' + 4(h + 1/4) = 0.$$

The general solution to this equation is of the form

$$h(\theta) = A \cos 2\theta + B \sin 2\theta - 1/4.$$

The symmetry of h about 0 in $[-\alpha, \alpha]$ implies that $B = 0$, and the initial conditions impose that $A = (1/4) \cos \alpha$. Therefore,

$$h(\theta) = \frac{-1 + \cos 2\theta}{4 \cos 2\alpha}.$$

Because this holds for all θ between $-\alpha/2$ and $\alpha/2$, and $h > 0$, we must have $\cos 2\alpha > 0$ and with the same sign as $\cos 2\theta$. This says that $\alpha < \pi/4$, and in this case the solution f exists.

3 Domain in the hyperbolic plane

Represent the hyperbolic plane as the set (x, y) with $x > 0$, endowed with the metric tensor

$$\frac{1}{x^2}(dx \otimes dx + dy \otimes dy).$$

The laplacian $\triangle_h$ corresponding to this metric tensor is

$$\triangle_h f = x^2 \triangle_e f = x^2(f_{xx} + f_{yy})$$

and the equation to solve is:

$$f_{xx} + f_{yy} = -x^{-2}$$

with the pertinent initial conditions.

3.1 Neighborhoods of geodesics

The first domain to consider is a neighborhood of a geodesic. As all are isometric, it is enough to consider the geodesic $y = 0$. A tubular neighborhood is precisely a euclidean cone which in polar coordiantes we take as (r, θ) with $\theta \in [-\alpha, \alpha]$, and $\alpha < \pi/2$.

As in the previous example, we see that composition with the isometry $(x, y) \mapsto (\lambda x, \lambda y)$, leaves the solution f invariant. That is $f(r, \theta) = f(\theta)$ is a solution to

$$f''(\theta) = \frac{-1}{\cos^2 \theta}$$

and so

$$f(\theta) = \log(\cos\theta / \cos\alpha).$$

3.2 Horoballs

Consider now an horoball. We may assume that it is symmetric with respect to $y = 0$ and tangent to the point at infinity. The domain is the set of points (x, y) with $x \geq a > 0$. As vertical translations are isometries with leave the boundary of this horoball invariant, the solution f of $\triangle_e f = -x^{-2}$ is invariant under this isometries. That is, $f(x + t, y) = f(x, y)$ for all t. Thus $f(x, y) = f(x)$ dependes only on the horizontal coordiante x. This says that f satisfies the equation

$$f''(x) = -x^{-2}$$

in $[a, \infty)$, with $f(a) = 0$. The minimal solution is

$$f(x) = \log(x/a).$$

3.3 Neighborhood of ideal point

The last kind of domain is a neighborhood of a point in the circle at infinity of $\mathbb{U}$. Such domain, up to isometry, is of the form $D = \{(x, y) \mid y \geq 0\}$. It contains arbitrarely thick neighborhoods of geodesics, hence no such function f exits, that is $f \equiv \infty$.

References

[1] Alberto Candel and Lawrence Conlon, *Foliations II*, Graduate Studies in Mathematics, Vol. 60, Amer. Math. Soc., Providence, RI, 2003.

[2] Eugene Dynkin, *Markov Processes*. Grundlehren der mathematischen Wissenschaften Vols. **121**, **122**, Springer, Berlin (1965).

[3] Étienne Ghys, *Topologie des feuilles génériques*. Ann. Math. **141** (1995), pp. 387–422