¶ **1.** The number pad of your calculator or your cellphone can be used to play a game between two players. Number pads for telephones are usually opposite way up from those of calculators, but that does not make any difference for playing this game.

$\boxed{7}$	$\boxed{8}$	$\boxed{9}$
$\boxed{4}$	$\boxed{5}$	$\boxed{6}$
$\boxed{1}$	$\boxed{2}$	$\boxed{3}$

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
$\boxed{4}$	$\boxed{5}$	$\boxed{6}$
$\boxed{7}$	$\boxed{8}$	$\boxed{9}$

The first player, Fred, turns on the calculator, presses a digit key and then presses the $\boxed{+}$ key. The second player, Sylvia, presses a digit key in the same row or column as the key just pressed by Fred, but different from the key that Fred has pressed, and the the $\boxed{+}$ key. The game proceed with Fred and Sylvia taking turns alternatively. The first player to reach a sum over 30 losses the game.

¶ **2.** Here is a sample game, played between Fred (first player) and Sylvia (second player):

- Fred presses $\boxed{8}$
- Sylvia could now press any of $\boxed{2}$, $\boxed{5}$ (same column as $\boxed{8}$), or $\boxed{7}$ or $\boxed{9}$ (same row as $\boxed{8}$). She presses $\boxed{7}$. The sum is $8 + 7 = 15$.
- It is Fred's turn. He has available the number keys $\boxed{4}$ $\boxed{1}$ $\boxed{9}$ and also $\boxed{8}$. He presses $\boxed{4}$. The total sum is $8 + 7 + 4 = 19$.
- Sylvia then presses $\boxed{6}$, bringing the total sum to $8 + 7 + 4 + 6 = 25$.
- Fred has the following number keys available: $\boxed{3}$, $\boxed{4}$, $\boxed{5}$ and $\boxed{9}$. Pressing $\boxed{9}$ brings the total sum to $25 + 9 = 34$, more than 30, making that a loss. He can press any of the other keys to prolong the game, but pressing $\boxed{5}$ bring the total sum to $25 + 5 = 30$ exactly.
- Since the total sum is currently 30, no matter what key Sylvia presses brings the total sum to more than 30, and thus she losses the game.

¶ 3. The number-pad Game is a game of perfect information. Each of the players know at each point in the game, what moves have been made prior to that point, what moves the current player can made, and what moves the opponent will be able to made in response to any possible move by the current player.
The game is also finite. Because the game ends when the total sum is over 30, after at most 20 moves one of the player will be declared a winner.

¶ 4. In most games, at his or her turn, a player may have to choose one of several moves. A **strategy** is a rule or decision making plan that tells the player which choice to make at teach point of the game.
For example, a strategy for the number pad game could be "Press the key directly to the right of that just pressed by your opponent (if this was the rightmost key on its row, then circle around to the beginning of the row)"

¶ 5. A strategy for a player that enables that player to win no matter what moves the opponents makes is called winning strategy for that player.
Is the strategy for the number key "Press the key directly to the right of that just pressed by your opponent (if this was the rightmost key on its row, then circle around to the beginning of the row)" a winning strategy?

¶ 6. Some games may end in a draw, like chess: the game ends with no winner and with no loser. If that is the case, then we may also speak of **drawing strategy** for a player as an strategy which even if it does not warranty a win for that player, it warranties that that player will not lose.

¶ 7. Thus, for a game between two players F(irst) and S(econd) we may ask: does F have a winning strategy? If not, does S have a winning strategy? If the answer to either question is yes, then we try to analyze the game and determine that strategy. It is not possible that both players have winning strategies, but it may be possible that neither of them does.
It is a mathematical theorem that in a game between two players, that id finite and with perfect information either:

(a) one of the players has a winning strategy, or

(b) both players have drawing strategies.

¶ 8. Because the number pad game cannot end in a draw, one of the players has a winning strategy. To find that strategy we need to analyze the game. One way to do so consists on describing the *game tree*, a directed tree that represents of all the ways a game can be played. Each node of the tree represents a position in the game, with a label indicating the player who can move from that node. At each node in the tree, a legal move is an arrow from that node to another node of the resulting position. Nodes that are terminal nodes (so no arrow issues from them) are labeled with the outcome of the game (win, loss, draw). Game trees are typically very large, so finding a winning strategy this way can be tedious, if not impossible because of time and space limitations.

¶ **9.** For the Number-Pad game with the full pad, the first player has a winning strategy, although it is harder to discover.

We modify the number-pad game to be played on a reduced number pad $\boxed{\begin{array}{cc} \boxed{1} & \boxed{2} \\ \boxed{3} & \boxed{4} \end{array}}$ and declaring a player to be the losing player if he or she is the first one to obtain a total sum greater than 7. We construct the game tree and then a wining strategy for one of the players.

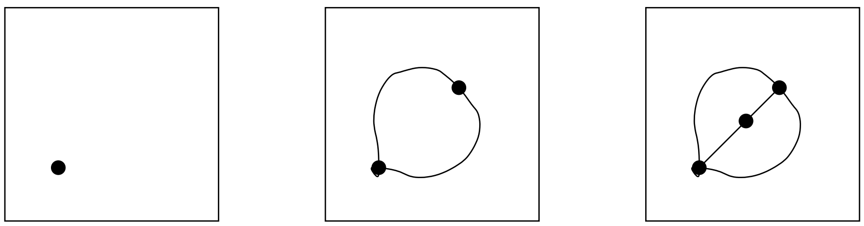
¶ **10.** **Sprouts** is a game played on a sheet of paper. It starts by placing n spots on it. A move consists on drawing a line that joins a spot to another one or to itself, and then placing a new spot anywhere along that line. The following rules must be observed:

(a) The line can have any shape, but it may not cross itself, cross a previously drawn line or pass through a previously drawn spot.

(b) No spot may have more than three lines emanating from it.

Players take turns drawing lines. The winner is the last player able to make a move.

¶ **11.** Sprouts can be played starting with just one spot. The first player only has the option of drawing a curve connecting the spot to itself. The second player then wins by joining the two spots.



Start of game F has moved S has moved, S wins

¶ **12.** Analyze the sprouts game with 2 initial dots. Does any of the players have a winning strategy?

¶ **13.** Games like Number-Pad and Sprouts are deterministic games: players take turns in a specific manner to choose among legal moves, the only possible outcomes involve one or more player winning and the other losing, or a draw (no one wins but no one loses), there is perfect information as to the moves available to each player at each point in the game, and players most prefer winning and least prefer losing. Those games are also non-cooperative: each player tries to get the best for him or herself: awin or a draw. In most real life games, to achieve the best possible outcome a player may have to form an alliance and cooperate with his or her opponent. The best illustration of a game of this kind is known as the Prisoner's dilemma.

> The police has arrested two suspected accomplices in a crime. The police has enough hard evidence to convict each suspect with a misdemeanor, but a felony conviction will require a confession. The suspects are put into separate interrogation rooms. Each suspect may either confess to the police, or remain quiet. If neither confesses, each will get a light sentence for the misdemeanor. If exactly one of the two confesses, prosecutors will make a deal with the confessor to receive probation only, and the other will receive a heavy sentence. If both confess, then each will receive a moderate sentence.

A startegy is dominated if it yields worse results that another strategy no matter what the other player does. The dilemma is caused by the tension between (1) choosing the strategy that will be always better for you regardless of the other player's choice, and (2) knowing that a better outcome might be possible if both players choose their dominated individual strategy.
If you are prisioner A, the you always do better by talking than by remaining quiet. If your accomplice talks, then you both get a moderate sentence, if your accomplice remains quiet, then you get probation. So "quiet" is your dominated strategy, and "talk" is the better choice for you regardless of what your accomplice does.. But if both prissioners use their best startegy, then both get a moderate sentence, while if both cooperate and use their domainated strategy "quite" both get a light sentence for misdemeanor.

Literature

[1] Fink and Guy, *The Number Pad Game* The College Math Journal, vol. 38 (2007)

[2] Gillman and Housman, *Models of Conflict and Cooperation*, AMS, 2009.

[3] Gardner, *Mathematical Carnival*, MAA, 1989.