

## Dido's Problem

As told by the Roman poet Virgil in his Aeneid (Book I), princess Dido, fleeing her murderous brother Pygmalion from the city of Tyre, landed in the shores of North Africa and offered to buy land from King Jarbas. She was offered the right to as much land as she could enclose with the hide of an ox. She accepted, cut the ox-hide into a very long thin strip and enclosed the maximum possible area by using the strip to mark the boundary of a semicircular region against the straight shoreline. There she founded the city of Carthage of which she became queen.

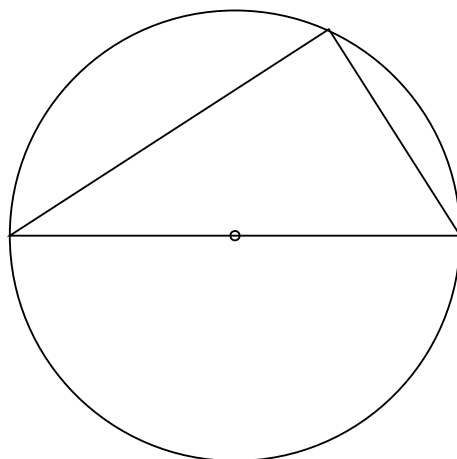
Dido made the most of the situation. First, she interpreted the work "enclose" as broadly as possible, thus cutting the ox-hide into very thin strips and making a very long strip.



*Dido Purchases Land for the Foundation of Carthage.* Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

¶ 1. Suppose that Dido's ox-hide was like a square about 8ft by 8ft, and that it was cut into thin strips of  $1/10$  inch wide and then tied together end to end to make a long thin strip. How many acres could Dido enclose with the resulting string?

¶ 2. To solve Dido's problem requires that you know this geometric fact: A right triangle hypotenuse is the diameter of its circumcircle. This is the converse to Thales' theorem: an inscribed triangle with the diameter as one of its sides is a right triangle.

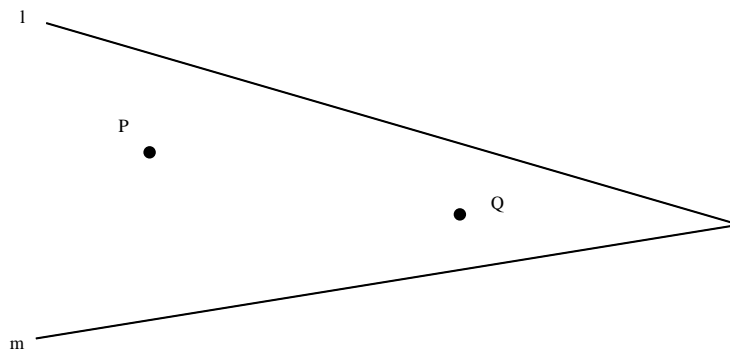


¶ 3. Given two segments of lengths  $a$  and  $b$ , find the triangle of maximum area having  $a$  and  $b$  as sides.

¶ 4. Given a line  $\ell$  and two points  $P$  and  $Q$  on the same side of  $\ell$ , for what point  $R$  on  $\ell$  is the distance  $PR + RQ$  shortest?



¶ 5. Given two points  $P$  and  $Q$  within two lines  $\ell$  and  $m$ , find the path of minimum length from  $P$ , then to  $\ell$ , then to  $m$  and then to  $Q$ . (Reflect  $Q$  on  $m$  to obtain  $Q'$  and then reflect  $Q'$  on  $\ell$  to get  $Q''$ .)



¶ 6. Given an area  $A$  and one side  $c = PQ$  of a triangle, find the triangle for which the sum of the two other side  $a + b$  is smallest.

¶ 7. Among all triangles with one side  $c$  and the sum of the two other sides  $a + b$  given, find that with largest area.

¶ 8. This set of problems utilize the so called geometric arithmetic inequality: for any positive numbers  $a$  and  $b$ , the geometric mean is never larger than the arithmetic mean:

$$\sqrt{ab} \leq \frac{a + b}{2}.$$

(a) Among all rectangles of given perimeter, find that of largest area.

(b) Among all rectangles of given area, find that of least perimeter.

¶ 9. Area and perimeter are quantities with dimension, hence not comparable. However, area scales like the square of the distance, so the quantities area and perimeter square are of the same dimension. Therefore for any planar figure  $F$ , the ratio

$$\mathcal{I}(F) = \frac{P^2}{A}$$

is a dimensionless quantity, called the isoperimetric ratio of the figure  $F$ .

(a) Find the isoperimetric ratio of an equilateral triangle.

(b) Find the isoperimetric ratio of a rectangle with sides of length  $a$  and  $b$ .

(c) What is the isoperimetric ratio of the region enclosed by circle?

¶ 10. Among all figures with the same perimeter, the one with largest area has the smallest isoperimetric constant. This is because if  $F$  and  $F'$  are figures with areas  $A \geq A'$  and equal perimeter  $P = P'$ , then

$$\mathfrak{I}(F) = \frac{p^2}{A} = \frac{p'^2}{A} \leq \frac{p^2}{A'} = \mathfrak{I}(F').$$

The solution to Dido's problem then establishes that the isoperimetric constant of any planar figure is  $\geq 4\pi$ , and is exactly  $4\pi$  for the circle. That is, among all figures with the same perimeter, a round disc encloses the largest area.

¶ 11. Among all polygons with the same number of sides, the regular polygon has the smallest isoperimetric constant. That is, among all polygons with the same number of sides and the same perimeter, the regular polygon encloses the largest area.