Tropical Math

Tropical Arithmetic

¶ 1. We define new operations on numbers:

$$x \oplus y = \min\{x, y\} \quad \text{and} \quad x \otimes y = x + y$$

For example:

$$5 \oplus 7 = 5 \qquad 5 \otimes 7 = 12$$

Find out the following:

(a) $6 \otimes 11$

(b) $(-6) \oplus (-2)$

(c) $(-3) \oplus 11$

(d) $3 \oplus (-11)$

(e) $(-3) \otimes 11$

(f) $3 \otimes (-11)$

¶ 2. Construct the (tropical) addition and multiplication tables for the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

$\oplus$	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

$\otimes$	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

¶ **3.** Most properties of usual arithmetic continue to hold in tropical arithmetic. For example, tropical addition and tropical multiplications are commutative:

(a) $x \oplus y = y \oplus x$

(b) $x \otimes y = y \otimes x$

¶ **4.** The distributive property continues to hold also in tropical arithmetic:

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$$

We write $(x \otimes y) \oplus (x \otimes z) = x \otimes y \oplus x \otimes z$ without parenthesis, with the convention that tropical products must be evaluated before tropical sums. Compute the following:

(a) $2 \otimes (6 \oplus 13)$

(b) $2 \otimes 6 \oplus 2 \otimes 13$

¶ **5.** The number 0 is the identity for tropical multiplication: $0 \otimes x = 0 + x = x$. But there is no number that is the identity element for tropical addition: if $x \oplus e = x$, for all x, then $x \leq e$ for all x. This suggests that we add ∞ to our arithmetic:

$$x \oplus \infty = \min\{x, \infty\} = x \quad \text{and} \quad x \otimes \infty = x + \infty = \infty.$$

In this way ∞ is the identity for $\oplus$ and 0 is the identity for $\otimes$.
Why can't we add both ∞ and $-\infty$ to this tropical arithmetic? what inconsistencies do you observe?

¶ **6.** Tropical arithmetic can be surprising. Prove that

$$(x \otimes x) \oplus (17 \otimes x) \oplus 2 = (x \oplus 1) \otimes (x \oplus 1)$$

¶ **7.** When it comes to subtraction, tropical arithmetic is tricky. In general, we cannot "subtract" in tropical arithmetic.
For example, "12 minus 7" will be the number x such that $7 \oplus x = 12$. But $7 \oplus x = \min\{7, x\} \leq 7 < 12$.
Show that you can do "a minus b" when $a \leq b$. That is, if $a \leq b$, find a number x such that $b \oplus x = a$.

¶ **8.** The Binomial Theorem is very easy to remember. Prove that $(x \oplus y)^n = x^n \oplus y^n$ (we simplify $x \otimes \cdots \otimes x = x^n$, that is:

$$(x \oplus y) \otimes \overset{n}{\cdots} \otimes (x \oplus y) = x \otimes \overset{n}{\cdots} \otimes x \oplus y \otimes \overset{n}{\cdots} \otimes y.$$

Try this out by verifying the following:

(a) $(x \oplus y)^2 = x^2 \oplus y^2$

(b) $(x \oplus y)^3 = x^3 \oplus y^3$

Polynomials

As in ordinary automatic, we have tropical polynomials in one or more variables. For example,

$$P(x) = 5 \otimes x \otimes x \oplus (-2) \otimes x \oplus 7$$

is a tropical polynomial in one variable, while

$$Q(x, y) = x \otimes x \oplus 3 \otimes x \otimes y \oplus (-2) \otimes x.$$

We examine their graphs.

¶ 9. Here is a polynomial function of degree 1 in one variable:

$$P(x) = [3 \otimes x] \oplus 5$$

which is the same as $\min(3 + x, 5)$.
To graph $y = [3 \otimes x] \oplus 5$, we plot the lower envelope of the lines $y = 3 + x$ and $y = 5$.

¶ 10. (a) Plot the graph of the degree 2 polynomial function $P(x) = [2 \otimes x \otimes x] \oplus [(-3) \otimes x] \oplus 1$

(b) Plot the graph of the degree 3 polynomial function $P(x) = [x \otimes x \otimes x] \oplus [(-1) \otimes x] \oplus 1$.

¶ 11. A line in the Euclidean plane is the graph of a polynomial of degree 1. In standard Euclidean plane geometry, two distinct lines are either parallel or intersect at exactly one point.
Describe the possible incidence relations for the graphs of two degree 1 tropical polynomials.

Tropical Geometry

¶ 12. A line in Euclidean geometry can also be described as solution sets of a linear equation in two variables $ax + by + c = 0$.
This suggests the correct definition of line in Tropical Geometry: we look at linear equations in two variables like $[a \otimes x] \oplus [b \otimes y] \oplus c$. This is the same as $\min\{x + a, y + b, c\}$.
The tropical line determined by $[a \otimes x] \oplus [b \otimes y] \oplus c$ is the collection of (x, y) where

$$\begin{cases} x + a = y + b \le c, & \text{or} \\ x + a = c \le y + b, & \text{or} \\ y + b = c \le x + a. \end{cases}$$

This is precisely the set of points (x, y) where the function $P : (x, y) \mapsto \min\{a + x, b + y, c\}$ fails to be linear.

¶ 13. Graph the lines determined by tropical polynomials of degree 1 in two variables given by:

(a) $P(x, y) = [3 \otimes x] \oplus [(-2) \otimes y] \oplus 1$

(b) $Q(x, y) = [(-2) \otimes x] \oplus [1 \otimes y] \oplus (-3)$

¶ 14. In a similar way, we can also graph more complicated graphs determined by tropical polynomials in two variables x and y. For example, the degree 2 polynomial

$$P(x, y) = [x \otimes x] \oplus [2 \otimes x \otimes y] \oplus [(-1) \otimes x] \oplus 1$$

is $P(x, y) = \min\{2x, 2 + x + y, x - 1, 1\}$, and it determines a tropical curve consisting of those points (x, y) where the minimum is reached at least twice, that is, two of $2x, 2 + x + y, x - 1, 1$ are equal and smaller than all others.

¶ **15** (Research Project)**.** Write a TI-84 program that, when imputing the coefficients of a tropical polynomial in two variables x and y, draws the graph of the tropical curve that it determines.

Literature

David Speyer and Bernd Sturmfels, *Tropical Mathematics*, Math. Mag. **82**(2009), pp. 163–173.