

Number trick Write any whole numbers of your choice against the numbers 1 and 2 in the right column of the following table:

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

Against 3, enter the sum of the entries against 1 and 2. Against 4, enter the sum of the entries against 2 and 3, the two numbers immediately above, and so on, until you reach the number 10.

Then the instructor asks the first question, and you will perform the calculations for the second question:

- (a) What is the number against 7?
- (b) What is the total sum?

How does the trick work?

n	u_n	$u_1 + \dots + u_n$
1	a	a
2	b	$a + b$
3	$a + b$	$2a + 2b$
4	$a + 2b$	$3a + 4b$
5	$2a + 3b$	
6	$3a + 5b$	
7	$5a + 8b$	
8	$8a + 13b$	
9	$13a + 21b$	
10	$21a + 34b$	$55a + 88b$
Total		

and the coefficients of b in that column are

$$1, 1, 2, 3, 5, 8, 13, 21, 34.$$

The coefficients of a in the right column are almost the same as those in the middle columns, except that they have shifted one space to the right

$$1, 1, 2, 3, 5, 8, 13, 21, 34.$$

and those of b are much like the above, except that they start in the third place and subtract 1:

$$0, 1, 2, 4, 7, 12, 20, 33, 54, 88.$$

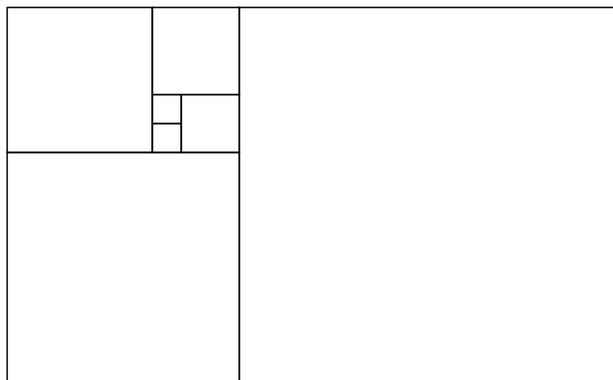
The coefficients of a in the middle column are

$$1, 0, 1, 1, 2, 3, 5, 8, 13, 21;$$

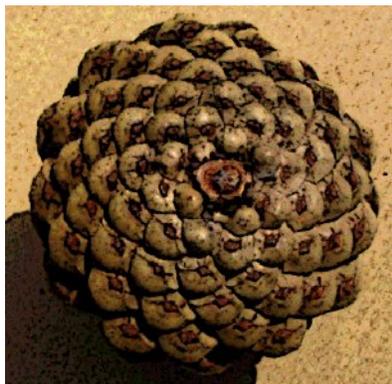
We this see that no matter which numbers a and b we start with, the total sum is 11 times the number against 7.

Spirals

¶ 2. Start with a square of unit side and construct a sort of spiral tiling of the plane as show in the figure. The sides of the squares increase as you make progress on the tiling. Write down the sequence of lengths of those squares.



¶ 3. The Fibonacci number arise naturally in the botanical phenomenon known as phyllotaxis (leaf arrangement). If you look at a flower like a daisy, or a broccoli floret, or a pine cone, or a pineapple, or a sunflower, you will notice certain spiral arrangement of their florets. Each has two interlocking spirals one whirling clockwise, the other counterclockwise. For example, the pineapple has 5 in one direction and 8 in the other direction, the pine cone has 8 spirals in one direction and 13 in the other direction, the daisy has 21 and 34, the sunflower has 55 and 89. It is always a consecutive pair of numbers in the Fibonacci sequence.



TI Program

Computing Fibonacci numbers turns out to be rather laborious. For example, it will take you a while to find out the number f_{100} . We will write a program for the TI-84 that will do that for us.

```
PROGRAM:FIBO
:Input 'TERM=', N
:1▶A
:1▶B
:2▶K
:While K<N
:A+B▶B
:B-A▶A
:K+1▶K
:End
:Disp B
```

```
PROGRAM:FIBOSQ
:Input 'TERM=', N
:1▶A
:1▶B
:2▶K
:While K<N
:Disp A
:Pause
:A+B▶B
:B-A▶A
:K+1▶K
:End
:Disp A
:Disp B
```

The program on the left inputs a whole number n and displays the term f_n of the Fibonacci sequence. The program on the right inputs a whole number n and displays all the terms f_1, f_2, \dots, f_n . The terms of the Fibonacci sequence grow quite rapidly. For example, the 100th term is $f_{100} = 354224848179261915075$ (21 digits). Your TI-84 cannot handle numbers this big, or can it?

¶ 4. The terms of the Fibonacci sequence satisfy the following formula

$$f_n = \frac{\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This impressive expression is called Binet's formula for f_n . How do we guess such a complicated expression for f_n ? The key to understand the Fibonacci sequence is to notice that it tries to be a geometric sequence; after all, it is quite related to spirals. You are familiar with these geometric sequences: for example

$$g_1 = 3, g_2 = 9, g_n = 3g_{n-1}$$

in which each terms is thrice the previous one, so the general term is $g_n = 3^n$, a power of a fixed number (3 in this case). The whole sequence is determined by the ratio of any two consecutive terms: $g_n/g_{n-1} = 3$ in this case, and then the general term g_n is the n th power of this ratio:

$$g_1 = 3^1, g_2 = 3^2, \dots, g_n = 3^n$$

For the Fibonacci sequence this is not quite true: there is no number r such that $f_n = r^n$. But it is almost true: if we compute the ratio f_n/f_{n-1} of any two consecutive terms, we notice that f_n/f_{n-1} is nearly the same for all n , especially if n is large. You may test that fact by filling out the table below.

$\frac{f_2}{f_1}$		$\frac{f_{21}}{f_{20}}$	
$\frac{f_3}{f_2}$		$\frac{f_{31}}{f_{30}}$	
$\frac{f_4}{f_3}$		$\frac{f_{41}}{f_{40}}$	
$\frac{f_5}{f_4}$		$\frac{f_{51}}{f_{50}}$	
$\frac{f_6}{f_5}$		$\frac{f_{101}}{f_{100}}$	
$\frac{f_7}{f_6}$		$\frac{f_{201}}{f_{200}}$	
$\frac{f_8}{f_7}$		$\frac{f_{501}}{f_{500}}$	
$\frac{f_9}{f_8}$		$\frac{f_{1001}}{f_{1000}}$	
$\frac{f_{10}}{f_9}$		$\frac{f_{10001}}{f_{10000}}$	
$\frac{f_{11}}{f_{10}}$		$\frac{f_{100001}}{f_{100000}}$	

The geometric sequence $g_n = 3^n$ satisfies a recurrence relation $g_n = 3g_{n-1}$, and so does the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$. Thus, we may try to see if there is a number r so that the sequence of powers r^n satisfies the Fibonacci recurrence relation: $f_n = f_{n-1} + f_{n-2}$, that is, $r^n = r^{n-1} + r^{n-2}$, or

$$r^2 = r + 1.$$

¶ 5. Find the solutions to the equation $r^2 = r + 1$. If you don't remember the quadratic formula, the following TI-84 Program allows you to solve the quadratic equation $ax^2 + bx + c = 0$.

```
PROGRAM:QUADRATIC
:Disp 'SOLVE AX^2+BX+C=0'
:Prompt A, B, C
:B^2-4*A*C>D
:If D>0
:Then
:Disp 'TWO REAL SOLUTIONS', (-B-√(D))/(2*A), (-B+√(D))/(2*A)
:End
:If D=0
:Then
:Disp 'ONE REAL SOLUTION', -B/(2*A)
:End
:If D<0
:Then
:Disp 'COMPLEX SOLUTIONS', (-B-√(abs(D))*i)/(2*A), (-B+√(abs(D))*i)/(2*A)
:End
```

The solutions to the equation $r^2 = r + 1$ are two very popular numbers. The largest is called the Golden Ratio $\phi = \frac{1 + \sqrt{5}}{2}$, and the other is $\psi = \frac{1 - \sqrt{5}}{2}$.

¶ 6. Verify the relation $\psi = -\frac{1}{\phi}$

¶ 7. To prove Binet's formula, we look for f_n in the form $f_n = A\phi^n + B\psi^n$, where A and B are two numbers to be determined. Since we want to have $f_1 = f_2 = 1$, we must have

$$\begin{aligned}A\phi + B\psi &= 1 \\A\phi^2 + B\psi^2 &= 1\end{aligned}$$

This is a system of two equations with two unknowns; you can easily find that $A = -B = \frac{1}{\sqrt{5}}$, and thus we obtain Binet's formula for f_n :

$$f_n = \frac{\phi^n - \psi^n}{\phi - \psi}.$$

¶ 8. It turns out that the the numbers in the Fibonacci sequence are related in a variety of ways. Here are some of these relations for you to confirm.

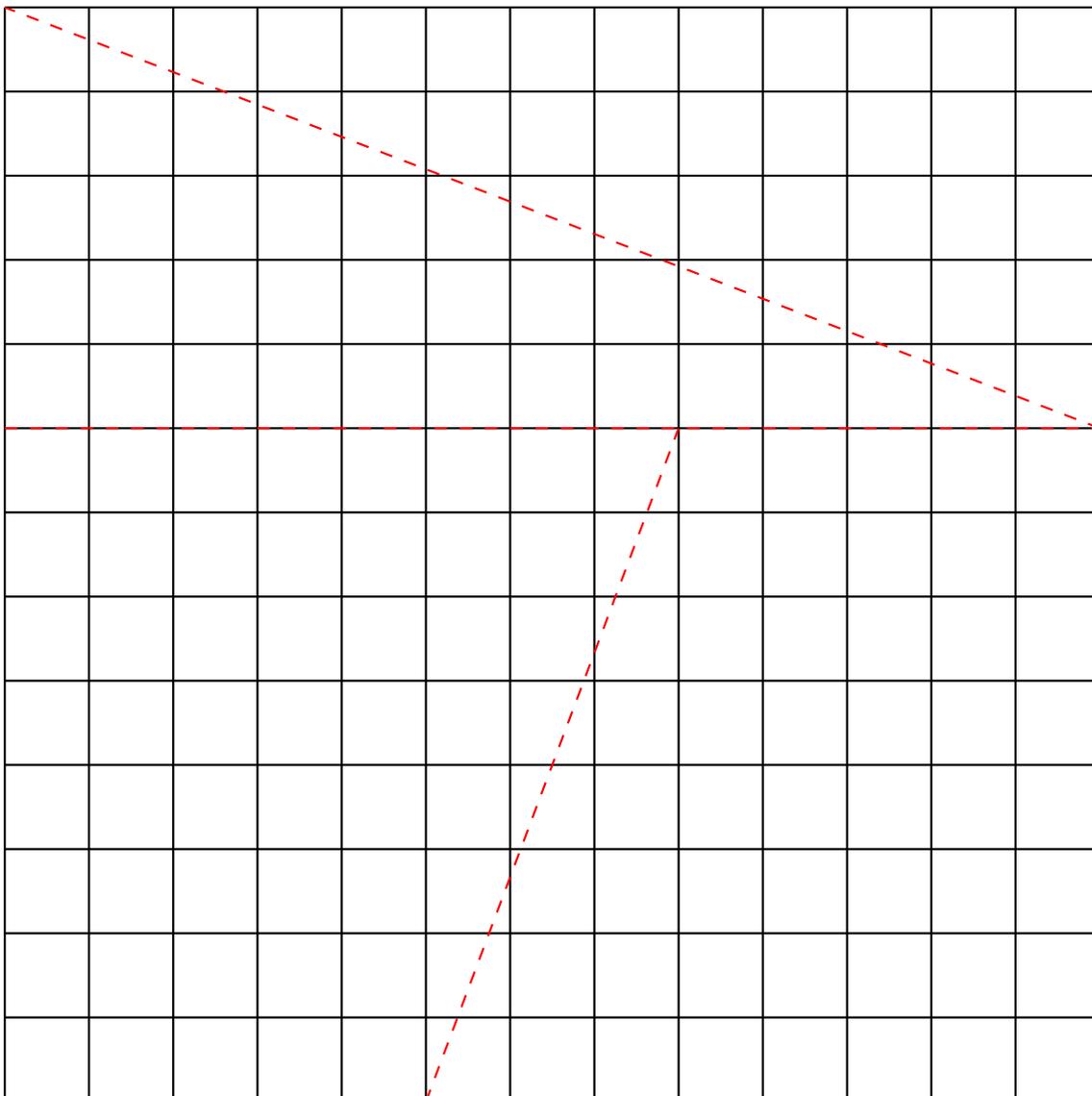
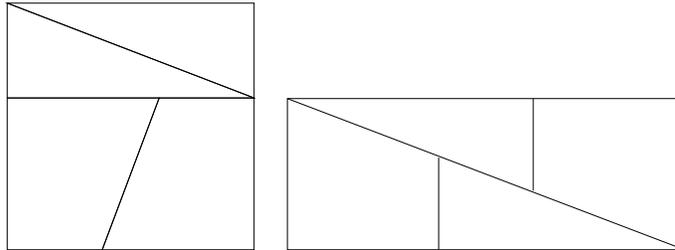
(a) $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$ (Simson, 1753).

(b) $f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$ (Lucas, 1876).

(c) $f_{2n+1} = f_{n+1}^2 + f_n^2$, $f_{2n} = f_{n+1}^2 - f_n^2$ (Lucas, 1876).

¶ 9. If you know about Taylor series, polynomials, and their coefficients, from your calculus class, you may want to compute them for the function $\frac{1}{1-x-x^2}$ at $x = 0$.

¶ 10. The relation $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for the Fibonacci numbers obtained in ¶8 affords the following geometric curiosity. Cut the 13×13 square below along the dotted lines, and reassemble the 4 pieces as indicated in the small diagram to obtain a 21×8 rectangle. Wait! $13 \times 13 = 169$ but $21 \times 8 = 168$! where did the little square go?

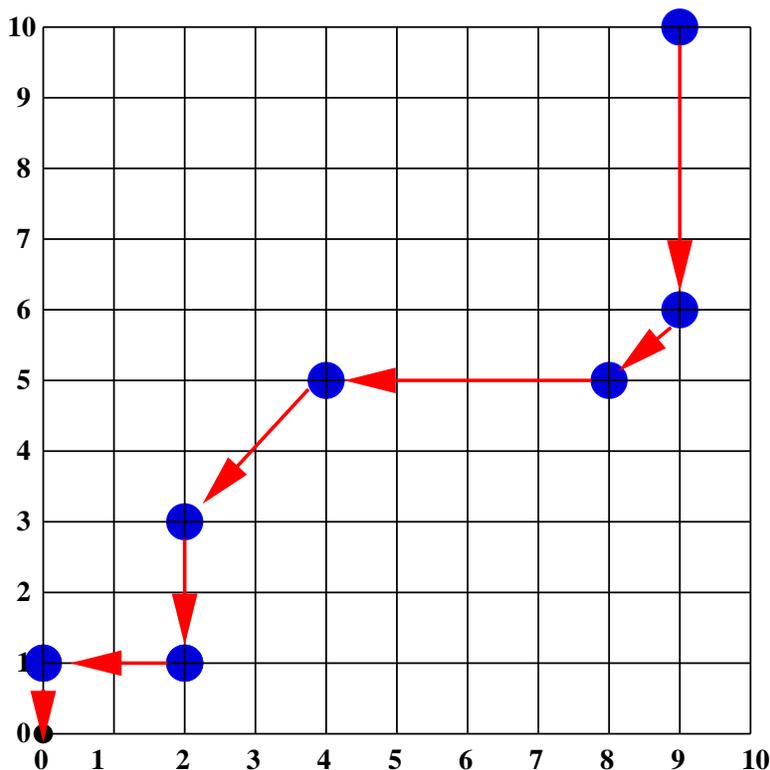


A game of no chance

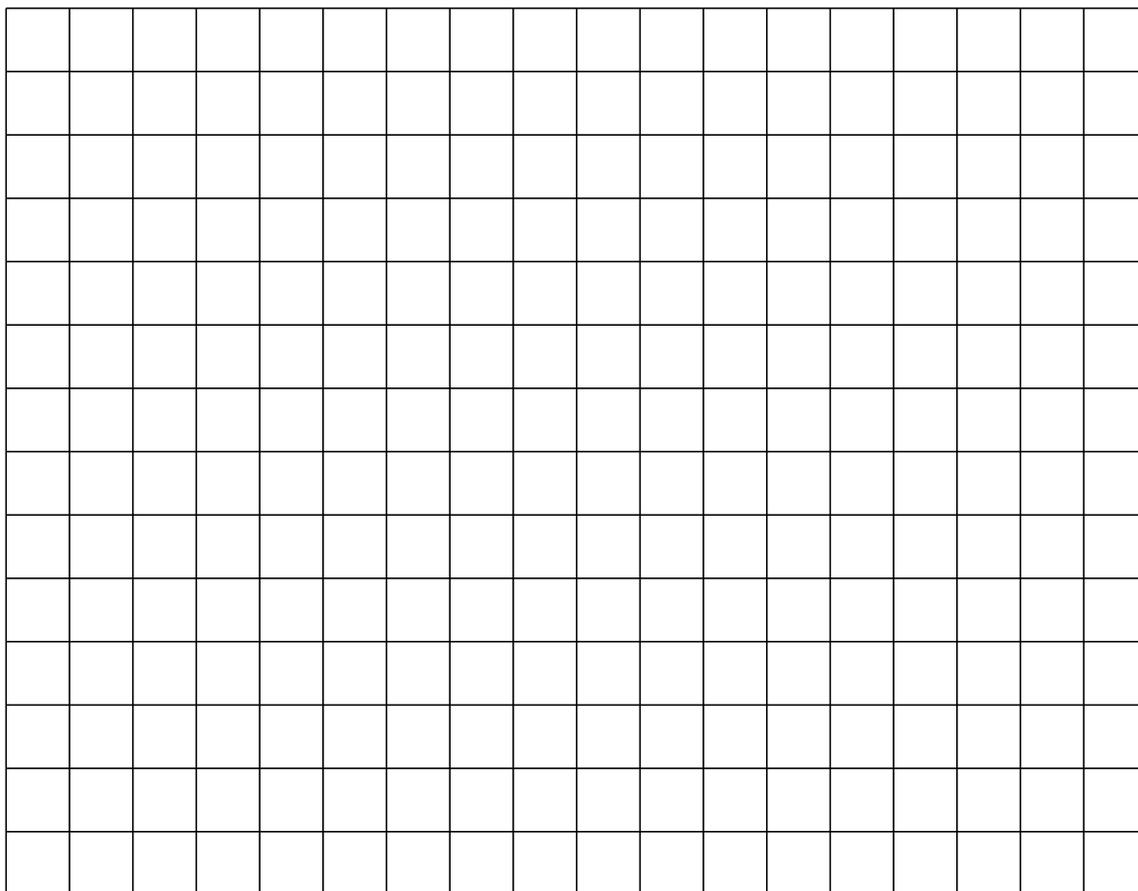
¶ 11. Here is a game (called Wythoff's game) to play when you go out for pizza with a friend. Each of you takes out all the change in your pockets and puts it on a pile on the table. There are now two piles of coins, most likely having different number of coins (if not, move one coin from one pile to the other). The game is played as follows. Each of you, in turn, removes certain amount of coins from the piles. You can do that in three different ways: remove any amount from your pile, or any amount from your friend's pile, or from both piles at once (if you do this, the same amount of coins must be removed from both piles at once). The winner is the one that takes the last coin.

If you don't have change readily available, you may play this game on a board like the one below. You place a token on any of the crossings and then you and your friend move the token in turns. The valid moves are: straight to the left, or downright south, or diagonally. The winner is the first to reach the bottom left corner of the board. If you think of this grid as a piece of the first quadrant of the Cartesian plane, you may record a game by writing down the coordinates of the point where the token is placed like thus: the token is placed by your friend at (9, 10), you move it to (9, 6), then your friend moves it to (8, 5), then you to (4, 5), then your friend moves it diagonally to (2, 3), then you move it south to (2, 1), then your friend moves it left to (0, 1), and then you move it to (0, 0) for the win, like so.

$$(9, 10) \xrightarrow{\text{You}} (9, 6) \xrightarrow{\text{Friend}} (8, 5) \xrightarrow{\text{You}} (4, 5) \xrightarrow{\text{Friend}} (2, 3) \xrightarrow{\text{You}} (2, 1) \xrightarrow{\text{Friend}} (0, 1) \xrightarrow{\text{You}} (0, 0)$$



¶ 12. In fact, once a player has placed the token at $(2, 1)$, then that player is guaranteed a win, no matter what his or her opponent does. For this reason, the place labeled $(2, 1)$ is called a safe place for that player. Clearly, $(1, 2)$ is also a safe place. Can you find other safe places? What about $(3, 5)$? (Use the grid below.)



¶ 13. Fill in this table below with as many (X, Y) coordinates of safe places as you can find. Is there any pattern?

X																			
Y																			

Literature

- H.S.M. Coxeter, *The Golden section. Phyllotaxis, and Wythoff's game*. Scripta Math. **19** (1953), pp 135–143
- D'Arcy Thompson, *On Growth and Form*, Dover.