Chapter 4

Applications

4.1 Singularities

Let f be analytic on $U \setminus \{z_0\}$, where U is an open subset of $\mathbf{C}$ and $z_0 \in U$. In this case z_0 is said to be an isolated singularity of f. The purpose of this section is to determine the behavior of f near z_0.

Theorem 4.1.1 (Cauchy's Formula for an Annulus). *Let f be analytic on an open set U containing the annulus $r_1 \le |z - z_0| \le r_2$, where $0 < r_1 < r_2 < \infty$. Let $\Gamma_i = |z - z_0| = r_i$, $i = 1, 2$, oriented counterclockwise. Then for $r_1 < |z - z_0| < r_2$,*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_2 - \Gamma_1} \frac{f(w)}{w - z}\, dw.$$

Proof. By Cauchy's Integral Formula,

$$f(z) \operatorname{ind}(\Gamma_2 - \Gamma_1; z_0) = \frac{1}{2\pi i} \int_{\Gamma_2 - \Gamma_1} \frac{f(w)}{w - z}\, dw.$$

By the properties of the index, if $r_1 < |z - z_0| < r_2$,

$$\operatorname{ind}(\Gamma_2 - \Gamma_1; z) = \operatorname{ind}(\Gamma_2; z) - \operatorname{ind}(\Gamma_1; z) = 1 - 0 = 1.$$

□

Theorem 4.1.2 (Laurent Series). *Let f be analytic on $U = \{s_1 < |z - z_0| < s_2\}$, where $0 \le s_1 < s_2 \le \infty$. Then f is given by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad z \in U,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{n+1}}\, dw$$

and Γ is any circle of radius r, $s_1 < r < s_2$, and center z_0. The series converges absolutely on U, and converges uniformly on compact subsets.

Proof. Choose $s_1 < r_1 < r_2 < s_2$, and let Γ_1 and Γ_2 be the circles $|w - z_0| = r_1$ and $|w - z_0| = r_2$, respectively. By Cauchy's Integral Formula for an Annulus (Theorem 4.1.1) applied to a point z such that $r_1 < |z - z_0| < r_2$,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z}\, dw.$$

If $w \in \Gamma_2$, then

$$\left| \frac{z - z_0}{w - z_0} \right| = \frac{|z - z_0|}{r_2} < 1,$$

so the series

$$\frac{1}{w - z} = \frac{1}{(w - z_0)\left(1 - \dfrac{z - z_0}{w - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

converges absolutely and uniformly on compact subsets in $|w - z_0| < r_2$, hence uniformly on Γ_2. By multiplying by the bounded function $\frac{1}{2\pi i} f(w)$ (which preserves uniform convergence) and integrating terms by term, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z}\, dw = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{(w - z_0)^{n-1}}\, dw, \qquad n = 0, 1, 2, \cdots$$

The second integral is treated similarly. If $w \in \Gamma_1$, then

$$\left| \frac{w - z_0}{z - z_0} \right| \leq \frac{r_1}{|z - z_0|} < 1$$

and the series

$$\frac{-1}{w - z} = \sum_{n=1}^{\infty} \frac{(w - z_0)^{n-1}}{(z - z_0)^n}$$

converges uniformly on Γ_2. Multiplying by $\frac{1}{2\pi i} f(w)$ and integrating term by term,

$$-\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z}\, dw = \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$b_n = \frac{1}{2\pi i} \int_{\Gamma_2} f(w)(w - z_0)^{n-1}\, dw, \qquad n = 1, 2, \cdots$$

Replacing the index $n = 1, 2, \cdot$ above by $-n = -1, -2, \cdots$ and writing

$$a_n = b_{-n} = \frac{1}{2\pi i} \int_{\Gamma_1} f(w)(w - z_0)^{-n-1}\, dw$$

for $n = -1, -2, \cdots$ we obtain the expansion

$$-\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{w-z}\,dw = \sum_{n=-1}^{-\infty} a_n(z-z_0)^n.$$

□

Example 4.1.3. Replacing z by $1/z$ in the power series

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \cdots \qquad |z| < \infty$$

we have the Laurent series

$$e^{1/z} = 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots \qquad 0 < |z| < \infty$$

Note that the series contains no positive powers of z, and that it has an infinite number of negative powers.

Note that the coefficient $a_{-1} = 1$; and according to Laurent's Series theorem, that coefficient is given by

$$a_{-1} = \frac{1}{2\pi i}\int_\Gamma e^{1/z}\,dz$$

where Γ is any positive oriented circle at the origin. Therefore,

$$\int_\Gamma e^{1/z}\,dz = 2\pi i.$$

This method of evaluating certain integrals will be developed further in subsequent sections.

Example 4.1.4. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series where $z_0 = i$. That is

$$f(z) = \sum_{-\infty}^{\infty} a_n(z-i)^n, \qquad 0 < |z-i| < \infty,$$

where $a_{-2} = 1$ and all other coefficients are zero. From Laurent series theorem we conclude that if Γ is a positively oriented circle $|z-i| = r$, then

$$\int_\Gamma \frac{1}{(z-i)^{n+2}}\,dz = \begin{cases} 0, & n \neq -2 \\ 2\pi i, & n = -2 \end{cases}$$

Example 4.1.5. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

is analytic on $\mathbf{C} \setminus \{1, 2\}$. It is analytic on the annuli $A_1 = \{|z| < 1\}$, $A_2 = \{1 < |z| < 2\}$ and $A_3 = \{2 < |z| < \infty\}$, and it has a Laurent series on each of them. Their representation can be fund with the help of the representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1.$$

To find the representation on A_1, we write

$$f(z) = -\frac{1}{1-z} + \frac{1}{2}\left(\frac{1}{1-(z/2)}\right)$$

and note that $|z| < 1$ and $|z/2| < 1$ on A_1,

$$f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n = \sum_{n=0}^{\infty} (2^{-1-n} - 1) z^n$$

for $|z| < 1$.

As for the representation on A_2, we write

$$f(z) = \frac{1}{z}\left(\frac{1}{1-(1/z)}\right) + \frac{1}{2}\left(\frac{1}{1-(z/2)}\right).$$

Since $|1/z| < 1$ and $|z/2| < 1$ if $1 < |z| < 2$, it follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad 1 < |z| < 2.$$

By replacing n by $n-1$ in the first series, and then interchange them, we obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n + \sum_{n=-\infty}^{-1} z^n \qquad 1 < |z| < 2.$$

This must be the Laurent series for f on the annulus A_2 because there is at most one such series.

The representation in A_2 is obtained in a similar way. First write

$$f(z) = \frac{1}{z}\left(\frac{1}{1-(1/z)}\right) + \frac{1}{z}\left(\frac{1}{1-(2/z)}\right),$$

and note that if $2 < |z| < \infty$, then $|1/z| < 1$ and $|2/z| < 1$. Therefore,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}, \qquad 2 < |z| < \infty$$

That is

$$f(z) = \sum_{n=-\infty}^{1} (1 - 2^{n-1}) z^n, \qquad 2 < |z| < \infty.$$

Remark. The coefficient a_n of the Laurent expansion of f is not necessarily $\dfrac{f^{(n)}(z_0)}{n!}$ for $n \geq 0$, because f is only analytic on $s_1 < |z - z_0| < s_2$ and may not have an analytic extension to $|z - z_0| < s_2$.

Theorem 4.1.6. *Suppose that* $f(z) = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$ *on* $U = \{s_1 < |z-z_0| < s_2\}$, $0 \le s_1 < s_2 \le \infty$. *Then* b_n *is given by*

$$b_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw.$$

That is, the Laurent expansion of f *on* U *is unique.*

Proof. The power series representing f converges uniformly on compact subsets of U. Multiply both sides by $\dfrac{1}{(z-z_0)^{k+1}}$ and integrate over Γ to obtain

$$\begin{aligned}\int_\Gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz &= \sum_{n=-\infty}^{\infty} b_n \int_\Gamma (z-z_0)^{n-k-1}\,dz \\ &= b_k\end{aligned}$$

by the argument in **??**. ☐

Definition 4.1.7. Let f have an isolated singularity at z_0, so that f has a Laurent expansion about z_0 of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

valid for $0 < |z-z_0| < r$, for some $r > 0$.

The sum of the negative powers of the Laurent series, that is, $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ is called the *principal part* of the Laurent expansion of f about z_0.

If the Laurent expansion contains no negative powers of $(z-z_0)$, then f is said to have a removable singularity at z_0. in this case, f can be extended to z_0 by setting $f(z_0) = a_0$.

If the principal part has finitely many non-zero term, that is, if

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + \sum_{n=1}^{\infty} a_n(z-z_0)^n$$

and $a_{-m} \ne 0$, then f is said to have a pole of order m at z_0 (a simple pole if $m = 1$).

In this case $(z-z_0)^m f(z)$ has a removable singularity at z_0, and $\lim_{z\to z_0}(z-z_0)^m f(z) = a_{-m} \ne 0$. In this case, setting $f(z_0) = \infty$ we obtain an analytic mapping of U into the Riemann sphere.

Finally, if the principal part of the Laurent series of f about z_0 contains infinitely many non-zero terms, then f is said to have an isolated essential singularity at z_0.

Lemma 4.1.8. *Let* f *have an isolated singularity at* z_0, *and let* $M(f, z_0, r) = \max\{|f(z)| \mid |z-z_0| = r\}$. *If there are constants* $k > 0$ *and* $\alpha \ge 0$ *such that* $M(f, z_0, r) \le kr^{-\alpha}$ *for all sufficiently small* $r > 0$, *then* f *has either a removable singularity at* z_0 *or a pole of order* $\le \alpha$.

Proof. The coefficient a_{-n} of the Laurent series can be estimated by using the integral representation

$$|a_{-n}| \le kr^{n-\alpha}$$

which converges to 0 as $r \to 0$ if $n > \alpha$; thus $a_{-n} = 0$ in this situation. ☐

Theorem 4.1.9 (Casorati-Weierstrass Theorem). *Let f have an essential singularity at z_0. Then for any $r > 0$, the image of the punctured disk $0 < |z - z_0| < r$ is dense in* $\mathbf{C}$.

Proof. Justifying the thesis is equivalent to proving that for any complex number w, the function $g(z) = \dfrac{1}{f(z) - w}$ is unbounded in any deleted neighborhood of z_0.

Assume that g is bounded on $V = \{0 < |z - z_0| < b\}$. In particular, $f(z) \neq w$ for all $z \in V$, hence g is analytic on V. Now $M(g, z_0, r) \leq K$, for some constant K and all $0 < r < b$, by assumption. It follows from Lemma 4.1.8 (with $\alpha = 0$) that g has a removable singularity at z_0. But on V, $f(z) = w + \dfrac{1}{g(z)}$; therefore, if m is the order of the zero of g at z_0 (setting $m = 0$ if $g(z_0) \neq 0$), then $(z - z_0)^m f(z)$ has a removable singularity at z_0. Consequently,

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n, \qquad z \in V$$

and it follows, after dividing by $(z - z_0)^m$, that f has either a removable singularity at z_0 or a pole of order m at z_0, contradicting the hypothesis. □

Theorem 4.1.10 (Classification of Singularities). *Let f have an isolated singularity at z_0.*

(a) There is a removable singularity at z_0 if and only if $f(z)$ approaches a finite limit as $z \to z_0$.

(b) There is a pole of order m at z_0 ($m = 1, 2 \cdots$) if and only if $(z - z_0)^m$ approaches a finite non-zero limit as $z \to z_0$, and in this case $f(z) \to \infty$ as $z \to z_0$.

(c) There is an essential singularity at z_0 if and only if $f(z)$ does not approach a finite or infinite limit as $z \to z_0$.

Proof. (a) The "only if" part follows from Definition 4.1.7; the "if"part follows from Theorem 2.2.11.

(b) The "only if" part follows from Definition 4.1.7; for the "if" part, note that if $(z - z_0)^m f(z)$ approaches a finite non-zero limit, then $(z - z_0)^m f(z)$ has a removable singularity at z_0 by (a), hence is given by $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ with $b_0 \neq 0$. Thus, by dividing by $(z - z_0)^m$, it follows that f has a pole of order m at z_0.

(c) The "only if" part follows from Casorati-Weierstrass Theorem (Theorem 4.1.9); the "if" part from (a) and (b). □

The behavior of a complex function f at ∞ may be studied by considering the function $g(z) = f(1/z)$ at $z = 0$.

4.2 Meromorphic Functions

Definition 4.2.1. The function f has an isolated singularity at ∞ if and only if f is analytic on a set $|z| > r$ and the function $g(z) = f(1/z)$ has an isolated singularity at $z = 0$. Removable singularities, poles, and essential singularities at ∞ are defined similarly.

Definition 4.2.2. A function f on an open subset of the Riemann sphere $\mathbf{P}$ is meromorphic on U if it is analytic on U except for poles and removable singularities.

Example 4.2.3. Let $R(z) = P(z)/Q(z)$ be a rational function, where P and Q are polynomials. The R is a meromorphic function on $\mathbf{P}$.

Theorem 4.2.4. *If f is meromorphic on $\mathbf{P}$, then f is a rational function.*

4.3 Calculus of Residues

In this section we present a technique which allows for rapid evaluation of integrals $\int_\gamma f$ where γ is a cosed path in U and f is analytic on U except for isolated singularities.

Definition 4.3.1. If f has an isolated singularity at z_0, the coefficient a_{-1} of the Laurent expansion of f about z_0 is called the residue of f at z_0, and denoted by $\operatorname{res}(f; z_0)$.

Lemma 4.3.2. *Let f be analytic on $U \setminus \{z_0\}$. Let R be a rectangle whose closure is contained in U. If $z_0 \in R$, then*

$$\operatorname{res}(f; z_0) = \frac{1}{2\pi i} \int_{\partial R} f(z)\, dz.$$

Proof. We may replace ∂R by a circle Γ with center z_0, by the First Cauchy Theorem (Theorem 3.3.6. Then, Theorem 4.1.2,

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma f(z)\, dz &= \sum_{n=-\infty}^{\infty} \frac{a_n}{2\pi i} \int_\Gamma (z - z_0)^n\, dz \\ &= a_{-1} \end{aligned}$$

by the calculation in the proof of the Cauchy Integral Formula for a Circle (Theorem 2.2.7.) □

Lemma 4.3.3. *Let γ be a closed path in $\mathbf{C}$, S a subset of $\mathbf{C}$ whose closure is disjoint from γ^*. Assume that whenever w is a limit point of S, then $\operatorname{ind}(\gamma; w) = 0$. Then $\operatorname{ind}(\gamma; z) = 0$ for all but finitely many $z \in S$.*

Proof. The set A consisting of those z satisfying $\operatorname{ind}(\gamma; z) = 0$ is an open subset of $\mathbf{C} \setminus \gamma^*$ which contains $|z| > r$, for r sufficiently large. Therefore $\mathbf{C} \setminus A$ is compact. If infinitely many points of S belong to $\mathbf{C} \setminus A$, then S has a limit point in $\mathbf{C} \setminus A$, contradicting the hypothesis. □

Theorem 4.3.4 (Residue Theorem). *Let f be analytic on U except for isolated singularities at the points $w_1, w_2, \cdots$. Let γ be a closed path (or cycle) in U such that $\operatorname{ind}(\gamma; z) = 0$ for all z not in U, and such that none of the w_j belong to γ^*. Then*

$$\frac{1}{2\pi i} \int_\gamma f = \sum_j \operatorname{ind}(\gamma; w_j) \operatorname{res}(f; w_j).$$

Proof. First note that $\operatorname{ind}(\gamma; w_j) = 0$ for all but finitely many w_j, hence the sum in the statement is finite. Indeed, let $S = \{w_j\}$. If w is a limit point of S, then $w \notin U$ since all the singularities are isolated; thus $\operatorname{ind}(\gamma; w) = 0$. Moreover, the closure of S does not meet γ^* because neither S meets γ^* nor the limit point of S meet γ^*.

Let $w_1, \cdots, w_n$ be the singularities for which $\operatorname{ind}(\gamma; w_j) \neq 0$. It may be assumed that γ is a polygonal path with edges parallel to the coordinate axes and contained in $U \setminus \{w_1, w_2, \cdots\}$. By a slight modification of γ, if necessary, it may also be assumed that none of the $w_1, \cdots, w_n$ lie on the rectangular grid induced by the polygonal path.

By Lemma 3.3.4, the path γ is equivalent to a cycle of the form $\displaystyle\sum_{j=1}^{m} \operatorname{ind}(\gamma; z_j)\partial R_j$, and for each $k = 1, \cdots, n$, the singularity w_k lies in some R_j. The grid may be taken so fine that no two singularities among $\{w_1, \cdots, w_n\}$ lie in the same R_j.

Let $V = U \setminus \{w_{n+1}, \cdots\}$. If the closure of R_j is not contained in V, let $z_0 \in \overline{R_j} \setminus V$. Since $z_0 \notin V$, it cannot be in $\gamma^* \subset V$, and so the segment $[z_0, z_j]$ does not meet γ^*. It follows that z_0 and z_j lie in the same component of $\mathbf{C} \setminus \gamma^*$, and thus the indexes $\operatorname{ind}(\gamma; z_0) = \operatorname{ind}(\gamma; z_j)$. If $z_0 \not in U$, then $\operatorname{ind}(\gamma; z_0) = 0$ by hypothesis; if $z_0 = w_k$ for some $k > n$, then $\operatorname{ind}(\gamma; z_0) = 0$ by the present construction. In any case, γ is equivalent to a cycle of the form $\sigma = \displaystyle\sum_{j=1}^{r} \operatorname{ind}(\gamma; z_j)\partial R_j$, where each rectangle R_j has closure contained in V and none of $w_1, \cdots, w_n$ is in the boundary of R_j.

Therefore f is defined and continuous on σ^*, and the equivalence of σ and γ yields

$$\int_\gamma f = \sum_{j=1}^{r} \int_{\partial R_j} f.$$

Each of the rectangles appearing in this sum contains at most one of the singularities $w_1, \cdots, w_n$. If $w_k \in R_j$, for some $k = 1, \cdots, n$, then $\operatorname{ind}(\gamma; w_k) = \operatorname{ind}(\gamma; z_k)$ and $\int_{\partial R_k} f = 2\pi i \operatorname{res}(f; w_j)$. If R_j contains no w_k, then $\int_{\partial R_j} f = 0$. It follows that

$$\frac{1}{2\pi i} \int f = \sum_{j=1}^{n} \operatorname{ind}(\gamma; w_j) \operatorname{res}(f; w_j).$$

□

Example 4.3.5. Evaluate the integral $\int_\gamma \dfrac{\operatorname{Log} z}{1 + e^z}\, dz$, where γ is the curve:

The advantage of expressing an integral in terms of residues is that it is often possible to compute the residues. There is in fact an explicit formula for the residue at a pole.

Theorem 4.3.6 (Residue at a Pole). *If f has a pole of order m at z_0, then*

$$\operatorname{res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} (z - z_0) f(z) \right).$$

In particular, if z_0 is a simple pole,

$$\operatorname{res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof. Multiply the Laurent expansion

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

by $(z-z_0)^m$, differentiate $m-1$ times, and take the limit as $z \to z_0$. □

Lemma 4.3.7. *Let f be analytic at z_0 and have a zero of order m there. Then f'/f has a simple pole at z_0 with* $\operatorname{res}(f'/f; z_0) = m$.

Proof. Write $f(z) = (z-z_0)^m g(z)$, where g is analytic at z_0 and $z_0 \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)},$$

Since g'/g is analytic at z_0, the residue

$$\operatorname{res}(f'/f; z_0) = m.$$

□

Example 4.3.8. If f has a pole of order m at z_0, then f'/f has a simple pole at z_0 with residue $\operatorname{res}(f'/f; z_0) = -m$.

Theorem 4.3.9 (Argument Principle). *Let f be analytic on the connected open set $U \subset$ **C**, and let γ be a closed path in U such that f is never 0 on γ^* and such that* $\operatorname{ind}(\gamma; z) = 0$ *for every $z \notin U$. If $z_1, \cdots$ are the distinct zeros of f with multiplicities $m_1, \cdots$, then*

$$\operatorname{ind}(f \circ \gamma; 0) = \sum_j m_j \operatorname{ind}(\gamma; z_j).$$

Proof. This follows directly form Theorem 3.2.5. Indeed,

$$\begin{aligned}
\operatorname{ind}(f \circ \gamma; 0) &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz && \text{by Theorem 3.2.5} \\
&= \sum_j \operatorname{res}(f'/f; z_j) \operatorname{ind}(\gamma; z_j) && \text{by Residue Theorem} \\
&= \sum_j m_j \operatorname{ind}(\gamma; z_j) && \text{by Lemma 4.3.7}
\end{aligned}$$

□

Remark. Intuitively, the number of times that $f(z)$ winds about the origin as z traverses the path γ is the number of zeros of f inside γ^*, each zero counted according to its multiplicity and its index with respect to γ.

If U is not assumed to be connected, then we have to add the hypothesis that f is not identically zero on each component of U.

Example 4.3.10. Let $f(z) = (z-a)^m$ and let $\gamma(t) = e^{ikt}$, $0 \le t \le 2\pi$ and k an integer. Then f has a zero at a of multiplicity m. The index of a with respect to γ is 0 if $|a| > 1$ and k if $|a| < 1$. Therefore, $\operatorname{ind}(f \circ \gamma; 0) = mk$ if $|a| < 1$ and 0 otherwise.

Theorem 4.3.11 (Generalized Argument Principle). *Let f and g be analytic on the open set U, with neither f nor g identically zero on a component of U. Let γ be a closed path on U such that f and g are never 0 on γ^*, and such that* $\operatorname{ind}(\gamma; z) = 0$ *for all $z \notin U$. If $z_1, z_2, \cdots$ are the zeros of f with multiplicities $n_1, \cdots$, and $w_1, w_2, \cdots$ are the zeros of g with multiplicities $m_1, m_2, \cdots$, then*

$$\operatorname{ind}\left(\frac{f}{g} \circ \gamma; 0\right) = \sum_j n_j \operatorname{ind}(\gamma; z_j) - \sum_k m_k \operatorname{ind}(\gamma; w_k).$$

Proof. Theorem 4.3.9 implies that

$$\operatorname{ind}\left(\frac{f}{g} \circ \gamma; 0\right) = \frac{1}{2\pi i} \int_\gamma \frac{(f/g)'}{f/g}.$$

But

$$\frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}$$

and the result follows from the previous theorem. □

Example 4.3.12. Let $f(z) = \dfrac{(z-1)(z-3+4i)}{(z+2i)^2}$, and let γ be the circle of center 0 and radius 3. Find $\operatorname{ind}(f \circ \gamma; 0)$.

Solution. The numerator has simple zeros at $z = 1$ and $z = 3 - 4i$. The denominator has a zero of order 2 at $-2i$. The index of 1 and $-2i$ with respect to γ is 1, and the index of $3 - 4i$ is 0. The Generalized Argument Principle yields

$$\begin{aligned} \operatorname{ind}(f \circ \gamma; 0) &= 1 \cdot \operatorname{ind}(\gamma; 1) + 1 \cdot \operatorname{ind}(\gamma; 3-4i) - 2 \cdot \operatorname{ind}(\gamma; -2i) \\ &= 1 + 0 - 2 = -1 \end{aligned}$$

Theorem 4.3.13 (Rouche's Theorem). *Let f and g be analytic on the connected open set U. Suppose that f has zeros $z_1, \cdots$ with multiplicities $n_1, \cdots$ and g has zeros $w_1, \cdots$ with multiplicities $m_1, \cdots$. Let γ be a path in U such that* $\operatorname{ind}(\gamma; z) = 0$ *for all $z \notin U$. Assume also that $|f(z) - g(z)| < |f(z)|$ for all z in γ^*. Then* $\operatorname{ind}(f \circ \gamma; 0) = \operatorname{ind}(g \circ \gamma; 0)$*; hence*

$$\sum_j n_j \operatorname{ind}(\gamma; z_j) = \sum_k m_k \operatorname{ind}(\gamma; w_k)$$

Thus f and g have the same number of zeros inside γ^*, counting index and multiplicity.

Proof. The hypothesis that $|f - g| < |f|$ on γ^* implies that f and g are never 0 on γ^*. Therefore, f is never 0 in some neighborhood of γ^*. If $h = 1 + \dfrac{g-f}{f}$, then $g = hf$ and h is never 0 on γ^*. In fact, $|1 - h| < \delta$ on γ^*, so that the curve $h \circ \gamma$ is contained in the disk $D(1; 1)$.

Now

$$\frac{g'}{g} = \frac{f'}{f} + \frac{h'}{h}$$

so that by Theorem 3.2.5

$$\begin{aligned} \operatorname{ind}(g\circ\gamma;0) &= \frac{1}{2\pi i}\int_\gamma \frac{g'}{g} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} + \frac{1}{2\pi i}\int_\gamma \frac{h'}{h} \\ &= \operatorname{ind}(f\circ\gamma;0) + \operatorname{ind}(h\circ\gamma;0). \end{aligned}$$

But $h\circ\gamma$ is a closed path lying in the disk $D(1,1)$, and thus $\operatorname{ind}(h\circ\gamma;0)=0$ because that disk does no contain 0.

□

Example 4.3.14. A polynomial of degree $n \geq 0$ has exactly n zeros, counting multiplicity.

Example 4.3.15. Rouche's theorem gives an easy proof of the following version of the maximum principle: if f is analytic at z_0, then there is z near z_0 such that $|f(z)| \geq |f(z_0)|$. For if $|f(z)| < |f(z_0)|$ in a disk $|z-z_0| \leq r$, then $f - f(z_0)$ and the constant $f(z_0)$ would have the same number of zeros inside $|z-z_0| = r$. Now $f - f(z_0)$ has a zero at z_0, and the only way that $f(z_0)$ can have a zero is if $f(z_0)=0$. This gives $|f(z)|<0$, a contradiction.

Example 4.3.16. Show that all the zeros of $p(z) = z^4+6z+1$ are inside the circle $|z|=2$. Moreover, three of the roots are in $1<|z|<2$.
Solution. The polynomial $q(z) = z^4+6z$ has roots at 0 and at $z=\sqrt[3]{-6}$. For these z, $1<|z|=\sqrt[3]{6}<2$ On $|z|=2$, $|z^4+6z+1| \geq 16-13 = 3 > |p(z)-q(z)| = 1$. The zeros of $q(z)$ all have index 1 with respect to $|z|=2$. The zeros of $p(z)$ have index either 0 or 1, and the sum of theirs multiplicities is 4.

On $|z|=1$, $|p(z)-q(z)|=1$, and $|q(z)| \geq |6z|-1 = 5$. Therefore q has exactly one root inside $|z|=1$. To show that there are three roots in $1<|z|<2$, look at $|z|=1+\varepsilon$ for small ε.

Example 4.3.17. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on $|z| \leq R$, and if $a_0 \neq 0$, then f cannot vanish on $|z| < \dfrac{|a_0|R}{M(R)+|a_0|}$, where $M(r) = \sup_{|z|=r} |f(z)|$.

Theorem 4.3.18. *Let $U \subset \mathbf{C}$ be a connected open set whose boundary is a finite collection Γ of simple closed paths. Let f be analytic on $\overline{U} = U\cup\Gamma$, except for a finite number of poles in U, and moreover, f is not zero on Γ. Then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'}{f} = N-P,$$

where N is the number of zeros of f in U and P is the number of poles of f in U, counted according to their multiplicity. Here Γ is the cycle obtained by giving each component γ of Γ the orientation that makes U to be locally on the left of γ.

4.4 Integrals

The Residue Theorem can be used for evaluating integrals.

Rational functions of sin and cos To evaluate

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where R is a rational function, we substitute $z = e^{i\theta}$ and note that

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

so that the integral becomes

$$-i \int_{|z|=1} R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] dz.$$

Example 4.4.1. If $a > b > 0$, then show that

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

Improper Integrals Cauchy's theorem can be applied to compute improper integrals, like

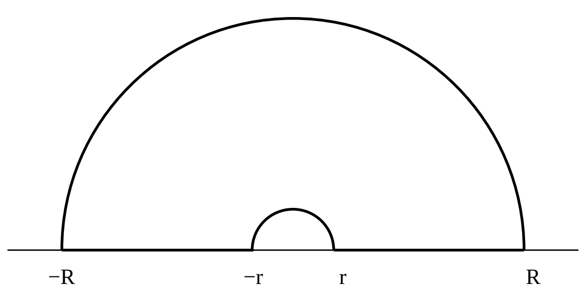
(1) $$\int_0^{\infty} \frac{\sin x}{x}\, dx$$

One defines the improper integrals of a real-valued function $f(x)$ of a real variable $x > 0$ which is bounded near 0 by

$$\int_0^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_0^R f(x)\, dx.$$

In general, f need not be absolutely integrable on $(0, \infty)$, as the example $f(x) = \sin x/x$ above.

To compute (1), consider the function $f(z) = e^{iz}/z$, which is analytic on $z \neq 0$. Let $R > r > 0$ be positive real numbers, and let γ be the closed path depicted below, oriented counterclockwise.



Then, since the index $\text{ind}(\gamma; 0) = 0$, Cauchy's Theorem implies that

$$\int_\gamma f(z) = 0$$

Parameterizing γ in the obvious way, this integral can be written as

$$(2)\quad \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{r}^{R} \frac{e^{ix}}{x}\,dx \\ - i\int_0^{\pi} e^{-r\sin\theta + ir\cos\theta}\,d\theta + i\int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\,d\theta.$$

Now

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{r}^{R} \frac{e^{ix}}{x}\,dx = 2i\int_0^R \frac{\sin x}{x}\,dx$$

and letting $r \to 0$, the third integral in (2) has the limit $-i\int_0^{\pi} d\theta = -\pi i$. Thus

$$2\int_0^R \frac{\sin x}{x}\,dx - \pi + \int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\,d\theta = 0.$$

Since this holds true for all $R > 0$, it also holds in the limit as $R \to \infty$. We now show that the last integral in (2) above approaches 0 as $R \to \infty$.

First note that

$$\left|e^{-R\sin\theta + iR\cos\theta}\right| = e^{-R\sin\theta},$$

so that

$$\left|\int_0^{\pi} e^{-R\sin\theta + iR\cos\theta}\,d\theta\right| \le \int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta.$$

If $0 \le \theta \le \pi/2$, then $2\theta/\pi \le \sin\theta$. Thus

$$2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \le 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{\pi}{R}(1 - e^{-R}).$$

This converges to 0 as $R \to \infty$, and therefore

$$\lim_{R\to\infty} 2\int_0^R \frac{\sin x}{x}\,dx - \pi = 0,$$

or

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Example 4.4.2. Evaluate the integral

$$\int_0^{\infty} \frac{1}{x^4 + a^4}\,dx,$$

where $a > 0$.

Solution. Since $1/(x^4 + a^4)$ is even,

$$\int_0^{\infty} \frac{1}{x^4 + a^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^4 + a^4}\,dx.$$

The function $f(z) = \dfrac{1}{z^4+a^4}$ has poles at $ae^{k\pi/4}$ $(k = 0, 1, 2, 3)$ of which $ae^{\pi/4}$ and $ae^{i3\pi/4}$ are in the upper half plane. Integrating along the boundary of a semicircle of radius R and center 0 in the upper half plane, noticing that $|f(x)| \le 1/R^4$ on $|z| = R$, and letting $R \to \infty$ we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx = \frac{1}{2}2\pi i\Big(\operatorname{res}(f; ae^{\pi i/4}) + \operatorname{res}(f; ae^{3\pi i/4})\Big) = \frac{\sqrt{2}\pi}{4a^3}.$$

□

4.5 The Open Mapping Theorem

The purpose of this section is to show that a non-constant analytic function maps open sets to open sets, and that a one-one analytic function has an analytic inverse. The next theorem examines the number of solutions of the equation $f(z) = w$, where w is fixed and z ranges over a neighborhood of a zero of f.

Theorem 4.5.1. *Let f be analytic and not identically constant on the disk $D(z_0, r)$, and assume that f has a zero of order k at z_0. Choose $r_1 < r$ so small that neither f nor f' is zero on $0 < |z - z_0| < r_1$, and let $m = \min\{|f(z)| \mid |z - z_0| = r_1\}$. If $0 < |w| < m$, then the equation $f(z) - w$ has exactly k solutions z in $D(z_0, r_1)$.*

Proof. Note that such r_1 must exists, for otherwise either f or f' has a limit point of zeros, hence f is identically constant on $D(z_0, r)$.

Let $\gamma(t) = z_0 + r_1 e^{it}$, $0 \le t \le 2\pi$; Then $|f(z)| \ge m > |w|$ on γ^*. Apply Rouche's Theorem to f and $g = f - w$ to obtain

$$\operatorname{ind}(f \circ \gamma; 0) = \operatorname{ind}((f - w) \circ \gamma; 0).$$

By hypothesis, f has a single 0 inside γ^* (that is, in $|z - z_0| < r_1$) of multiplicity k. By the Argument Principle,

$$\operatorname{ind}((f - w) \circ \gamma; 0) = \sum_j k_j \operatorname{ind}(\gamma; z_j)$$

where the z_j are the zeros of $f - w$ inside γ^*, and k_j is theirs respective multiplicity.

Because $\operatorname{ind}(\gamma; z_j) = 1$, if there are fewer than k such points, then $k_j > 1$ for at least one index j, and therefor $f - w$ has a zero of order greater than 1 at z_j. This inn turn implies that $f'(z_j) = 0$. This leads to a contradiction because the only point inside γ^* where $f' = 0$ is at z_0, and $f(z_0) - w = w \ne 0$. □

Theorem 4.5.2. *Let f be analytic and not identically constant on a disk $D(z_0, r)$ and suppose that f has a zero of order k at z_0. Then there is an open set $U \subset D(z_0, r)$, with $z_0 \in U$, such that for each $z \in U$, $z \ne z_0$,*

(a) $f(z) \ne 0$, and

(b) there are exactly k points z' in $U \setminus \{z_0\}$ such that $f(z') = f(z)$.

Proof. Let r_1 and m be as in previous Theorem 4.5.1. Let $U = D(z_0, r_1) \cap f^{-1}(D(0, m))$. If $z \in U \setminus \{z_0\}$, then $f(z) \neq 0$, by the choice of r_1, and $|f(z)| < m$ by the choice of U. Theorem 4.5.1 implies that there are exactly k points z' in $D(z_0, r_1)$ with $f(z') = f(z)$. Since $|f(z)| < m$, all such z' belong to $U \setminus \{z_0\}$. □

Corollary 4.5.3. *Let f be analytic at z_0. If $f'(z_0) \neq 0$, then there exists a neighborhood of z_0 on which f is one-one. If $f'(z_0) = 0$, then f cannot be one-one in any neighborhood of z_0.*

Proof. Apply Theorem 4.5.2 to $f(z) - f(z_0)$; note that $f(z) - f(z_0)$ has a zero of order 1 at $z = z_0$ if $f'(z_0) \neq 0$, and a zero of order > 1 at $z = z_0$ if $f'(z_0) = 0$. □

Theorem 4.5.4 (Open Mapping Theorem). *Let f be analytic on the open set $U \subset \mathbf{C}$, and not identically constant on any component of U. Then $f : U \to \mathbf{C}$ is an open mapping.*

Proof. It must be proved that if V is an open subset of U, then $f(V)$ is an open subset of $\mathbf{C}$. Let $z_0 \in V$ and define $g(z) = f(z) - f(z_0)$. Let $D(z_0; r) \subset V$ and construct r_1 and m as in Theorem 4.5.1. (Note that g is not identically constant on $D(z_0; r)$.) It follows from Theorem 4.5.1 that $D(0, m) \subset g(V)$. But then $D(f(z_0), m) \subset f(V)$, for if $|w - f(z_0)| < m$, then there is a point z in V such that $g(z) = w - f(z_0)$, and therefore $f(z) = w$. Thus $f(V)$ is open. □

Lemma 4.5.5. *Let U and V be open subset s of $\mathbf{C}$, f a one-one mapping of U onto V, with inverse g. Assume that (1) f is continuous, (2) g is differentiable, and (3) g' is never 0 on U. Then f is differentiable and $f' = \dfrac{1}{g' \circ f}$.*

Theorem 4.5.6. *Let f be analytic and one-one on an open set U. Then f^{-1} is analytic on the open set $f(U)$.*

Proof. Since f is one-one, it is not identically constant on any component of U, hence $f(U)$ is open by the Open Mapping Theorem. Also by the Open Mapping Theorem, $g = f^{-1}$ is continuous on $V = f(U)$. Because of Theorem 4.5.2, f' is never 0 on U. The result follows from the previous lemma. □

The next theorem gives a more explicit description of the nature of an analytic function near a zero of order k.

Theorem 4.5.7. *Let f be analytic and non-constant on U. Let $z_0 \in U$ and set $f(z_0) = w_0$. Let k be the order of the zero of $f - w_0$ at z_0. Then there exists a neighborhood V of z_0 in U, and analytic function φ on V such that*

(a) $f(z) = w_0 + (\varphi(z))^k$ for all z in V.

(b) φ' has no zero on V, and φ is a one-one mapping of V onto $D(0; r)$.

Proof. It may be assumed that U is a convex neighborhood of z_0, so small that $f(z) \neq w_0$ for all z in $U \setminus \{z_0\}$. Then there is an analytic function g on U such that

$$f(z) - w_0 = (z - z_0)^k g(z)$$

for all z in U and such that g is never zero on U. Therefore g'/g is analytic on U, and since U is convex, it has an analytic logarithm h on U. If $\varphi(z) = (z - z_0)\exp h(z)/k$, then (a) holds for all z in U.

Also $\varphi(z_0) = 0$ and, because this is a simple zero, $\varphi'(z_0) \neq 0$. The rest now follows at once from the Open mapping Theorem. □

Exercise 4.5.8. Let f be analytic at z_0, with $f(z_0) = 0$ and $f'(z_0) \neq 0$. Let $\Gamma = \{|z - z_0| = r\}$, where r is chosen small enough so that f is one-one on the disk $D(z_0; s)$ for some $s > r$. If h is analytic on $D(z_0; s)$ and $|w| < \min\{|f(z)| \mid z \in \Gamma\}$, then prove that

$$h(f^{-1}(w)) = \frac{1}{2\pi i}\int_\Gamma h(z)\frac{f'(z)}{f(z) - w}\,dz$$

In particular,

$$f^{-1}(w) = \frac{1}{2\pi i}\int_\Gamma z\frac{f'(z)}{f(z) - w}\,dz$$