Notes on Fourier Series

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This notes on Fourier series complement the textbook. Besides the textbook, other introductions to Fourier series (deeper but still elementary) are Chapter 8 of Courant-John [5] and Chapter 10 of Mardsen [6].

1 Introduction and terminology

We will be considering functions of a real variable with complex values. If $f : [a, b] \to \mathbf{C}$ is such function, then it can be written as

$$f = \Re f + i\Im f,$$

where $\Re f$ and $\Im f$ are its real and imaginary parts, respectively. We say that f is integrable on $[a, b]$ if both $\Re f$ and $\Im f$ are integrable there, in which case

$$\int_a^b f = \int_a^b \Re f + i \int_a^b \Im f.$$

A function or a real variable f is said to be **periodic** with period P if

$$f(x + P) = f(x)$$

holds for all x. Hence, if we know the values of f on an interval of length P, we know its values everywhere.

If f is a function defined on an interval $[a, b)$, we can extend f to a function defined for all x which is periodic of period $b - a$. We simply define $f(x)$ to be $f(x + n(b - a))$, where n is the integer such that $a \leq x + n(b - a) < b$.

The graph of a periodic function f has the same shape in any two consecutive intervals corresponding to a period. Thus, if P is the period of f,

$$\int_{-a}^{P-a} f = \int_0^P f,$$

for arbitrary number a.

In what follows we will usually consider functions of period 2π. They will be usually defined on $[0, 2\pi]$ or in $[-\pi, \pi]$. A function $f : [0, 2\pi] \to \mathbf{C}$ will be called continuous, differentiable, etc., if that is true for its periodic extension.

For instance, $f : [0, 2\pi] \to \mathbf{C}$ is continuous in this sense if f is continuous on $(0, 2\pi)$, right continuous at 0, left continuous at 2π, and $f(0) = f(2\pi)$.

We will denote by

$$f(x+) = \lim_{t \to 0} f(x + t^2)$$

the right limit of f at x, and by

$$f(x-) = \lim_{t \to 0} f(x - t^2)$$

the left limit at x.

The functions $x \mapsto e^{inx}$ (n an integer) have continuous derivatives of all orders on $[0, 2\pi]$. They will play a very important role in all what follows. They have the following orthogonality property:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{inx} e^{-imx} \, dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

Functions of the form

$$\sum_{n=-N}^{N} a_n e^{inx},$$

are usually called **trigonometric polynomials**. To see them more trigonometric you may use the identity

$$e^{inx} = \cos nx + i \sin nx,$$

which translates into

$$\sum_{n=-N}^{N} a_n e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx),$$

where $A_n = a_n + a_{-n}$ and $B_n = i(a_n - a_{-n})$.

We will usually consider periodic functions of period 2π. There is no particular reason for that. Another popular choice is period 1, and functions defined on $[0, 1]$. In this case, the basic functions are $x \mapsto e^{2\pi inx}$.

2 Fourier series

Suppose that f is an integrable function. Then

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{int} dt$$

is called the n-th **Fourier coefficient** of f. The **Fourier series** of f is

$$Sf(x) = \sum_{-\infty}^{\infty} \widehat{f}(n) e^{inx}.$$

If N is a positive integer, let $S_N f$ denote the function

$$S_N f(x) = \sum_{-N}^{N} \widehat{f}(n) e^{inx}.$$

The content of the next proposition is known as **Bessel's inequality.**

Proposition 1 *If $|f|^2$ is integrable, then*

$$\sum_{-N}^{N} |\widehat{f}(n)|^2 \le \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt.$$

Proof. We compute

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(t) - S_N f(t)|^2 dt &= \frac{1}{2\pi} \int_0^{2\pi} (f(t) - S_N f(t))\overline{(f(t) - S_N f(t))} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(|f(t)|^2 - f(t)\overline{S_N f(t)} - \overline{f(t)} S_N f(t) + |S_N f(t)|^2 \right) dt. \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{S_N f(t)}\, dt &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{-N}^{N} f(t) \overline{\widehat{f}(n) e^{int}}\, dt \\ &= \sum_{-N}^{N} \overline{\widehat{f}(n)} \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt \\ &= \sum_{-N}^{N} |\widehat{f}(n)|^2. \end{aligned}$$

Similarly

$$\frac{1}{2\pi} \sum_{-N}^{N} \int_0^{2\pi} \overline{f(t)} \widehat{f}(n) e^{int}\, dt = \sum_{-N}^{N} |\widehat{f}(n)|^2,$$

and a similar calculation shows that

$$\frac{1}{2\pi} \int_0^{2\pi} |S_N f(t)|^2\, dt = \sum_{-N}^{N} |\widehat{f}(n)|^2$$

also. Thus the original equation reduces to

$$\frac{1}{2\pi} \int_0^{2\pi} |f(t) - S_N f(t)|^2\, dt = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2\, dt - \sum_{-N}^{N} |\widehat{f}(n)|^2.$$

Since the left side is nonnegative, the proposition follows. □

Corollary 1 *With the same hypothesis on f, the Fourier coefficients $\widehat{f}(n) \to 0$ as $|n| \to \infty$.*

Proof. The proposition says that the series

$$\sum_{-\infty}^{\infty} |\widehat{f}(n)^2| = \lim_{N\to\infty} \sum_{-N}^{N} |\widehat{f}(n)|^2$$

converges, hence $|\widehat{f}(n)| \to 0$ as $|n| \to \infty$. □

3 Convergence of Fourier series

For each positive integer N, let

$$D_N(t) = \sum_{-N}^{N} e^{int}.$$

This function is continuous and periodic with period 2π. Note also that

$$\frac{1}{2\pi}\int_0^{2\pi} D_N(t)dt = 1.$$

The function D_N is called the **Dirichlet kernel**.

Proposition 2 *If t is not an integer multiple of 2π, then*

$$D_N(t) = \frac{e^{i(N+1)t} - e^{-iNt}}{e^{it} - 1} = \frac{\sin(N + \frac{1}{2})t}{\sin\frac{t}{2}}.$$

Proof. To obtain the first equality, note that $e^{it} = 1$ if and only if t is an integer multiple of 2π. Hence

$$(e^{it} - 1)D_N(t) = e^{i(N+1)t} - e^{-iNt}.$$

The second follows by multiplying and dividing the second expression by $e^{it/2}$, together with $2i\sin z = e^{iz} - e^{-iz}$. □

Proposition 3 *If f is integrable, then*

$$S_N f(x) = \frac{1}{2\pi}\int_0^{2\pi} D_N(t)f(x - t)dt.$$

Proof. We have

$$\begin{aligned} S_N f(x) &= \sum_{-N}^{N} \widehat{f}(n) e^{inx} \\ &= \sum_{-N}^{N} \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt \right) e^{inx} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \left(\sum_{-N}^{N} e^{in(x-t)} \right) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) D_N(x-t) dt. \end{aligned}$$

A change of variable $x - t = s$ finishes the proof. □

We say that a function $f : [0, 2\pi] \to \mathbf{C}$ satisfies a **Lipschitz condition** if there is a positive constant M such that the periodic extension of f satisfies

$$|f(s) - f(t)| \leq M|s - t|,$$

for all real numbers s, t.

In terms of the function f defined in $[0, 2\pi]$, this can be expressed as

$$|f(s) - f(t)| \leq M \min\{|s-t|, |s-t-2\pi|, |s-t+2\pi|\},$$

for all $s, t \in [0, 2\pi]$. The 'min' is the distance from s to t modulo 2π, and so it is the distance as measured on a circle of length 2π.

Proposition 4 *If f satisfies a Lipschitz condition, then $S_N f \to f$ uniformly as $N \to \infty$.*

Proof. We have

$$\begin{aligned} S_N f(x) - f(x) &= \frac{1}{2\pi} \int_0^{2\pi} (f(x-t) - f(x)) D_N(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) D_N(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) \frac{\sin(N+\frac{1}{2})t}{\sin(t/2)}\, dt \end{aligned}$$

The last equality follows from the periodicity of the integrated functions. Hence,

$$\begin{aligned} |S_N f(x) - f(x)| \leq\ & \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) \frac{\cos(t/2)}{\sin(t/2)} \sin Nt\, dt \right| \\ & + \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) \cos Nt\, dt \right|. \end{aligned}$$

Let

$$h(t) = (f(x-t) - f(t))\frac{\cos(t/2)}{\sin(t/2)}$$

for $t \neq 0$, and

$$k(t) = f(x-t) - f(x),$$

so that we have

$$|S_N f(x) - f(x)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} h(t)\sin Nt\,dt\right| + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} k(t)\cos Nt\,dt\right|.$$

The function

$$h(t) = \frac{f(x-t) - f(t)}{t}\frac{t/2}{\sin(t/2)}2\cos(t/2)$$

is continuous at all $t \neq 0$ in $[-\pi, \pi]$. Furthermore,

$$1 \leq \left|\frac{t/2}{\sin(t/2)}\right| \leq \frac{\pi}{2}$$

and it extends to a continuous function at $t = 0$ by giving it the value 1 there.

The Lipschitz condition on f says that

$$|f(x-t) - f(x)| \leq M|t|.$$

Hence the function $|h(t)|$ is continuous and bounded on $[-\pi, 0)$ and on $(0, \pi]$. Thus $|h|^2$ is integrable on $[-\pi, \pi]$, and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)\sin Nt\,dt = \frac{\widehat{h}(N) - \widehat{h}(-N)}{2i}.$$

The function $|k|^2$ is integrable in $[0, 2\pi]$, and we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)\cos Nt\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)\frac{e^{iNt} + e^{-iNt}}{2}\,dt \\ &= \frac{\widehat{k}(N) + \widehat{k}(-N)}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}|S_N f(x) - f(x)| &\leq \left|\frac{\widehat{h}(N) - \widehat{h}(-N)}{2i}\right| + \left|\frac{\widehat{k}(N) + \widehat{k}(-N)}{2}\right| \\ &\leq \frac{1}{2}\left(|\widehat{h}(N)| + |\widehat{h}(-N))| + |\widehat{k}(N)| + |\widehat{k}(-N)|\right).\end{aligned}$$

and note that by the corollary to Bessel's inequality, the last term converges to 0 as $N \to \infty$. □

The technique used in the proof above can also be used to show the following

Corollary 2 *If f is integrable in $[0, 2\pi]$ and differentiable at x_0, then*

$$S_N f(x_0) \to f(x_0),$$

as $N \to \infty$.

A function $f : [a, b] \to \mathbf{C}$ is said to be **sectionally continuous** on $[a, b]$ if it is continuous except for a finite number of points $x_0, x_1, \cdots, x_n$ where the right and left limits $f(x_i+)$ and $f(x_i-)$ both exist.

Riemann-Lebesgue Lemma. *If f is integrable on $[a, b]$, then*

$$\lim_{\alpha \to \infty} \int_a^b f(x) \sin \alpha x \, dx = 0,$$

and

$$\lim_{\alpha \to \infty} \int_a^b f(x) \cos \alpha x \, dx = 0.$$

The proof of this lemma is outlined in Exercise 15-26 of Spivak [7]. An easy proof can be obtained in case that f and f' are sectionally continuous functions in $[a, b]$ by using integration by parts.

Proposition 5 *Suppose that $f : [0, 2\pi] \to \mathbf{C}$ is sectionally continuous, has a jump discontinuity at x_0, and that the left and right derivatives $f'(x_0+)$ and $f'(x_0-)$ both exist. Then the Fourier series of f converges to $(f(x_0+) + f(x_0-))/2$ at x_0.*

4 Integration of Fourier series

The following is a consequence of the fundamental theorem of calculus.

Proposition 6 *If $f : [a, b] \to \mathbf{C}$ is sectionally continuous, then*

$$F(x) = \int_a^x f(t) \, dt$$

is continuous. Furthermore, F is differentiable at each point of $[a, b]$, except perhaps at the points of discontinuity of f, where it has right and left derivatives.

Suppose that $f : [0, 2\pi] \to \mathbf{C}$ is sectionally continuous. Then it is integrable on $[0, 2\pi]$, so that we can compute its Fourier series. Let this be

$$Sf(t) = \sum_{-\infty}^{\infty} \widehat{f}(n) e^{int}.$$

Let

$$F(x) = \int_0^x (f(t) - \widehat{f}(0)) \, dt.$$

Then

$$F(0) = F(2\pi) = 0,$$

and together with the proposition above it follows that F satisfies a Lipschitz condition on $[0, 2\pi]$. Hence the Fourier series of F converges uniformly to the function F. To compute it we use integration by parts. If $n \neq 0$,

$$\widehat{F}(n) = \frac{-1}{in}\widehat{f}(n).$$

Note that you have to justify the validity of the integration by parts because F may not have continuous derivative on $[0, 2\pi]$.

Hence,

$$F(x) = \widehat{F}(0) + \sum_{n\neq 0} \frac{-1}{in}\widehat{f}(n)e^{inx}.$$

5 Weierstrass approximation theorem

Weierstrass approximation theorem says that a continuous function f defined on a closed interval $[a, b]$ is the uniform limit of a sequence of polynomials. We present a proof using the theory of Fourier series that we have developed (see also Exercise 3.4.6 of Adams-Guillemin [1]).

Note that this statement is independent of the size and position of the closed interval $[a, b]$, and thus we may assume that $[a, b]$ is contained in the open interval $(0, 2\pi)$. The first step in the proof is to approximate f by a piecewise linear function with any prescribed degree of accuracy $\varepsilon > 0$. This can be done because f is uniformly continuous (p. 143), that is, given $\varepsilon > 0$, there exists a partition $x_0, x_1, \cdots, x_n$ of the interval $[a, b]$ such that $|f(x) - f(y)| < \varepsilon/3$ if both x, y are in one of the subintervals $[x_i, x_{i+1}]$. It follows that if l is the function whose graph consists of the segments joining the points $(x_i, f(x_i))$ with $(x_{i+1}, f(x_{i+1}))$, then

$$|f(x) - l(x)| < \varepsilon/3$$

for each x in $[a, b]$.

The next step is to extend l to a function on $[0, 2\pi]$ so that it becomes periodic. This is done by adding to the graph of l the segments from $(0, 0)$ to $(x_0, f(x_0))$ and from $(x_n, f(x_n))$ to $(2\pi, 0)$. This extended function, which we continue to call l, being piecewise linear and continuous, satisfies a Lipschitz condition. It follows that it may be uniformly approximated by its Fourier series. That is, there is N such that

$$|S_N l(x) - l(x)| < \varepsilon/3$$

for all x in $[0, 2\pi]$.

Finally, by using that $e^{inx} = \cos nx + i \sin nx$, we see that $S_N l$ is a finite linear combination of the functions $\sin nx$ and $\cos nx$, $|n| \leq N$. Since the trigonometric functions $\sin nx$ and $\cos nx$ can be uniformly approximated by their Taylor polynomials, we can find a polynomial $P(x)$ such that

$$|P(x) - S_N l(x)| < \varepsilon/3$$

in $[0, 2\pi]$. Combining the three inequalities, we have

$$|P(x) - f(x)| < \varepsilon$$

in the smaller interval $[a, b]$.

6 Applications to number theory

The first application concerns the nature of the values of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at the even positive integers. It is adapted from Courant-John [5].

The Bernoulli polynomials $\varphi_n(x)$, $0 \leq x \leq 2\pi$, are defined recursively by the following relations:

$$\varphi_0(x) = 1,$$

$$\varphi_n'(x) = n\varphi_{n-1}(x),$$

and

$$\int_0^1 \varphi_n(x)\, dx = 0$$

for $n > 0$. That is, knowing φ_{n-1} we can calculate φ_n up to a constant, which is determined by the last condition. We see by induction that φ_n is a polynomial of degree n whose coefficients are rational numbers.

For $n > 1$ we have

$$\varphi_n(1) - \varphi_n(0) = \int_0^1 \varphi_n'(x)\, dx = 0.$$

Therefore, if we denote by $\psi_n(x)$ the periodic extension of the polynomial φ_n, then the functions ψ_n are continuous and satisfy a Lipschitz condition.

The function $\psi_1(x) = x - (1/2)$ is sectionally continuous. Its Fourier coefficients, as a function on $[0, 1]$, are:

$$\widehat{\psi}_1(n) = \int_0^1 \left(x - \frac{1}{2}\right) e^{2\pi i n x}\, dx = \begin{cases} 0, & \text{if } n = 0 \\ \dfrac{-1}{2\pi i n}, & \text{if } n \neq 0 \end{cases}$$

Therefore its Fourier series is

$$S\psi_1(x) = \frac{1}{2\pi} \sum_{n \neq 0} \frac{-1}{in} e^{2\pi i n x}.$$

The Fourier series for the other functions ψ_n are obtained by successively integrating this one. We obtain:

$$\psi_k(x) = \frac{k!}{(2\pi)^k} \sum_{n \neq 0} \frac{-1}{(in)^k} e^{2\pi i n x}.$$

In particular, if k is even,

$$\psi_k(x) = (-1)^{1+(k/2)} \frac{2(k!)}{(2\pi)^k} \sum_{n=1} \frac{1}{n^k} \cos 2\pi n x,$$

and if k is odd,

$$\psi_k(x) = (-1)^{(k+1)/2} \frac{2(k!)}{(2\pi)^k} \sum_{n=1} \frac{1}{n^k} \sin 2\pi n x,$$

These series ψ_k converge uniformly for all x and agree with φ_k in the interval $[0,1]$. We also see that $\psi_k(-x) = (-1)^n \psi_k(x)$,

Let

$$b_k = \varphi_k(0) = \begin{cases} -1/2, & \text{if } k = 1 \\ \psi_k(0), & \text{if } k > 1. \end{cases}$$

These are rational numbers, and from the Fourier series we see that

$$b_k = 0 \qquad \text{if } k \text{ is odd, } k \neq 1,$$

and

$$b_k = (-1)^{1+(k/2)} \frac{2(k!)}{(2\pi)^k} \sum_{n=1} \frac{1}{n^k}.$$

It follows that the values of the Riemann zeta function at the even integers are rational multiples of a power of (2π).

Not much is known about the values $\zeta(2m+1)$. Only recently Apéry [8] showed that $\zeta(3)$ is irrational.

The next application (Exercise 10.5 of Mardsen [6]) is a representation of the function $\sin \pi x$ as an infinite product that resembles the factorization of a polynomial. This was already known to Euler, of course by other means.

We consider the function $f(x) = \cos \lambda x$, $-\pi \leq x \leq \pi$, where λ is a non-integral real number. The function $f(-\pi) = f(\pi)$ so it can be extended to a periodic continuous function. To compute its Fourier series we use the interval $[-\pi, \pi]$. We have

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \lambda x \, e^{-inx} \, dx \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} (e^{i\lambda x} + e^{-i\lambda x}) e^{-inx} \, dx \\ &= \frac{(-1)^n}{2\pi} \frac{2\lambda}{\lambda^2 - n^2} \sin \lambda x. \end{aligned}$$

Since the function f satisfies a Lipschitz condition, its Fourier series converges to $f(x)$ at all points. Hence, for $-\pi \le x \le \pi$,

$$\cos \lambda x = \frac{\sin \lambda\pi}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{2\lambda}{\lambda^2 - n^2} e^{inx}.$$

In particular, for $x = \pi$ we have

$$\cos \pi\lambda = \frac{\sin \pi\lambda}{\pi} \left(\frac{1}{\lambda} + \sum_{n=1} \frac{2\lambda}{\lambda^2 - n^2} \right).$$

Therefore

$$\pi \tan \pi\lambda - \frac{1}{\lambda} = \sum_{n=1} \frac{2\lambda}{\lambda^2 - n^2},$$

if λ is not an integer. The series on the right converges uniformly for $0 \le \lambda \le \lambda_0 < 1$. The function on the left is integrable because $\pi \tan \pi\lambda - (1/\lambda) \to 0$ as $\lambda \to 0$. By integrating,

$$\log \left(\frac{\sin \pi\lambda}{\pi\lambda} \right) = \sum_{n=1} \log \left(1 - \frac{\lambda^2}{n^2} \right)$$

for $|\lambda| < 1$. That is

$$\sin \pi\lambda = \pi\lambda \prod_{n=1} \left(1 - \frac{\lambda^2}{n^2} \right).$$

These product formula is also valid for $\lambda = \pm 1$, and then for all real λ because the expression on the right defines a periodic function of period 2. This product formula is interesting because it exhibits directly that the function $\sin \pi\lambda$ vanishes at the integer values of λ. In this respect it corresponds to the factorization of a polynomial when its zeros are known.

If we take $\lambda = 1/2$, we obtain Wallis' product formula for $\pi/2$:

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9} \cdots .$$

7 The isoperimetric inequality

This application of Fourier series will show that among all simple closed plane curves of a given perimeter, the circle is the one that encloses the largest area.

By a plane curve we mean a continuous function $z : t \in [0, 2\pi] \mapsto z(t) \in \mathbf{C}$. It is closed if $z(0) = z(2\pi)$, and it is simple if the function z is one to one on $[0, 2\pi)$. We assume that the curves considered here have continuous derivative.

The length L of the curve $z(t)$, $0 \le t \le 2\pi$ is

$$L = \int_0^{2\pi} |z'(t)| \, dt$$

as is described in the Appendix to Chapter 13. We assume that $L = 2\pi$, and that the curve is parametrized by arc-length: $|z'(t)| = 1$ for all t. The area A enclosed by the curve z is seen to be

$$A = \frac{1}{2i}\int_0^{2\pi} \overline{z(t)} z'(t)\, dt.$$

The hypothesis imposed on the function z imply that it can be represented by its Fourier series:

$$z(t) = \sum_{-\infty}^{\infty} \widehat{z}(t) e^{int}.$$

By replacing $z(t)$ by $z(t) - \widehat{z}(0)$ (which is a translation of the plane, so it does not change the quantities A and L), we may assume that $\widehat{z}(0) = 0$.

The Fourier coefficients of $\overline{z}$ and z' are easily computed:

$$\widehat{\overline{z}}(n) = \frac{1}{2\pi}\int_0^{2\pi} \overline{z(t)} e^{-int}\, dt = \overline{\widehat{z}(-n)},$$

and

$$\widehat{z'}(n) = in\widehat{z}(n).$$

Hence

$$\begin{aligned} A &= \frac{1}{2i}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} \overline{\widehat{z}(-n)} e^{int} \sum_{m=-\infty}^{\infty} im\, \widehat{z}(m) e^{imt}\, dt \\ &= \pi \sum_{n\neq 0} n|\widehat{z}(n)|^2. \end{aligned}$$

On the other hand, for the length we have (recall the assumption $L = 2\pi$)

$$\begin{aligned} 2\pi &= \int_0^{2\pi} dt \\ &= \int_0^{2\pi} |z'(t)|^2\, dt \\ &= 2\pi \sum_{n\neq 0} n^2 |\widehat{z}(n)|^2. \end{aligned}$$

By comparing term by term we obtain the isoperimetric inequality

$$0 \leq \pi - A.$$

Equality holds if and only if

$$\sum_{n\neq 0} (n^2 - n)|\widehat{z}(n)|^2 = 0,$$

that is, if and only if $|\widehat{z}(n)|^2 = 0$ for $n = -1, \pm 2, \pm 3, \cdots$. In this situation,

$$z(t) = \widehat{z}(1) e^{it},$$

which is a circle of radius $|\widehat{z}(1)|$.

Proposition 7 *If $z(t)$ is a curve of length L enclosing an area A, then*

$$4\pi A \leq L^2,$$

with equality if and only if $z(t)$ is a circle.

Proof. The curve $w(t) = (2\pi/L)z(t)$ has lenght 2π and encloses an area of $(2\pi/L)^2 A$. By the discussion above,

$$0 \leq \pi - \left(\frac{2\pi}{L}\right)^2 A$$

or

$$4\pi A \leq L^2.$$

□

8 Ergodic theory

Ergodic theory is a field of mathematics which studies the long term average behavior of systems. The collection of all states of a system forms a space X. The evolution of the system is represented by a transformation $f : X \to X$, where $f(x)$ is taken to be the state at time 1 of a system which at time 0 is in the state x.

This of course sounds very abstract. Here is a concrete example. Suppose that your state space X is the unit circle $|z| = 1$ in the complex plane, and that your transformation is a rotation by an angle $2\pi\alpha$. Thus if you are at the point $z = e^{2\pi ix}$ at time 0, at time 1 you would be at the point $e^{2\pi i\alpha}z = e^{2\pi(x+\alpha)}$, and at $e^{2\pi(x+n\alpha)}$ at time n. If α is a rational number of the form p/q, then you would come back to the initial state at time q. On the other hand, if α is irrational, you will never visit any of the states through which you have already passed. In this case, we may further ask how the states are distributed on the circle.

Another problem related to this one is the following (see Arnold-Avez [3]). Let a_n, $n = 1, 2, \cdots$, denote the first digit of 2^n. Let $N(7, k)$ denote the number of 7's included in the first k terms of the sequence a_n. The problem is to compute, if it exists, the limit

$$\lim_{k\to\infty} \frac{N(7,k)}{k}.$$

Here are the first terms of the sequence; the number 7 is quite slow in showing up.

2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1,
2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 7, 1, 2, 5, 1, 2, 4, 9, 1,
3, 7, 1, 2, 5, $\cdots$

Let $x_1, x_2, \ldots$ be a sequence of numbers in the interval $[0,1)$. Given any pair of real numbers a, b, such that $0 \le a < b \le 1$, denote by $\xi_n(a,b)$ the number of the first n terms of the sequence x_k which are in the interval $[a,b)$.

A sequence x_k is said to be **uniformly distributed** in the interval $[0,1)$ if

$$\lim_{n\to\infty} \frac{\xi_n(a,b)}{n} = b - a$$

, for each pair of numbers $0 \le a < b \le 1$. For example, the sequence $x_k = 1 - (1/k)$ is not uniformly distributed in $[0,1)$.

Proposition 8 *An infinite sequence of numbers* (x_k) *in the interval* $[0,1)$ *is uniformly distributed if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_0^1 f(x)\,dx,$$

for every function f *integrable on* $[0,1]$.

Proof. Given an interval $[a,b)$, the function

$$\chi_{[a,b)}(x) = \begin{cases} 1, & \text{if } a \le x < b \\ 0, & \text{otherwise,} \end{cases}$$

is integrable on $[0,1)$. We have

$$\int_0^1 \chi_{[a,b)} = b - a,$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \chi_{[a,b)}(x_k) = \frac{\xi_n(a,b)}{n}.$$

Therefore the sequence is uniformly distributed.

Conversely, if the sequence is uniformly distributed, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_0^1 f(x)\,dx$$

holds when f is of the form $\chi_{[a,b)}$. Since any step function on $[0,1]$ is a linear combination of functions of the form $\chi_{[a,b)}$, this result also holds for them.

If f is integrable on $[0,1]$, then, by the theorem on p.256, given $\varepsilon > 0$, we can find step functions s_1, s_2 such that $s_1 \le f \le s_2$, and

$$\int_0^1 (s_2(x) - s_1(x))\,dx < \varepsilon.$$

Since the proposition holds for s_1,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_2(x_k) = \int_0^1 s_1(x)\,dx \le \varepsilon + \int_0^1 f(x)\,dx,$$

so that if n is large enough,

$$\frac{1}{n}\sum_{k=1}^{n} s_2(x_k) < 2\varepsilon + \int_0^1 f(x)\,dx.$$

Since $f \le s_2$, this implies

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) < 2\varepsilon + \int_0^1 f(x)\,dx,$$

for n sufficiently large. Similarly, by using s_1 instead of s_2, we obtain

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) > \int_0^1 f(x)\,dx - 2\varepsilon.$$

Therefore

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(x_k) - \int_0^1 f(x)\,dx\right| < 2\varepsilon,$$

for n large enough. □

Proposition 9 *A sequence x_k is uniformly distributed if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} e^{2\pi i m x_k} = 0,$$

for each non-zero integer m.

Proof. The function $x \mapsto e^{2\pi i m x}$ is integrable on $[0,1]$, and if $m \neq 0$

$$\int_0^1 e^{2\pi i m x}\,dx = 0.$$

Thus, if the sequence x_k is uniformly distributed, the conclusion holds by the previous proposition.

Conversely, suppose that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} e^{2\pi i m x_k} = 0,$$

for $m \neq 0$. Then, by linearity, the equation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_k) = \int_0^1 f(x)\,dx,$$

holds for every trigonometric polynomial (it obviously holds for the constant function $\equiv 1$).

It then follows that the equation also holds for every continuous periodic function on $[0,1]$, since the proof of the Weierstrass approximation theorem implies that they can be uniformly approximated by trigonometric polynomials. More precisely, given $\varepsilon > 0$, choose $P(x)$ so that

$$\sup_{0\le x\le 1} |f(x) - P(x)| < \varepsilon.$$

If we let $f_1 = P - \varepsilon$ and $f_2 = P + \varepsilon$, then $f_1 \le f \le f_2$ and

$$\int_0^1 (f_2(x) - f_1(x))\, dx = 2\varepsilon,$$

and the same argument as in the previous proposition shows that the equation also holds for continuous periodic functions on $[0,1]$.

If s is a step function on $[0,1]$, we can find continuous periodic functions f_1 and f_2 such that $f_1 \le s \le f_2$ and

$$\int_0^1 (f_2(x) - f_1(x))\, dx < \varepsilon.$$

The same argument then shows that the equation holds for step functions, and therefore it also holds for any integrable function on $[0,1]$. □

Proposition 10 *If λ is an irrational number, then the sequence of fractional parts $\{k\lambda\}$ is uniformly distributed in $[0,1)$.*

Proof. Clearly, $e^{2\pi in\{k\lambda\}} = e^{2\pi ink\lambda}$. For each integer $m \ne 0$, $m\lambda$ is an irrational number, and we have

$$\left|\sum_{k=0}^{n} e^{2\pi imk\alpha}\right| \le \frac{1}{|\sin \pi m\lambda|}.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi imk\lambda} = 0.$$

□

Further details on uniform distribution of numbes can be found in Chandrashekar [4].

Next we study the problem on the sequence of first digits of powers of 2. The first digit of 2^n is equal to k if and only if

$$k10^r \le 2^n < (k+1)10^r,$$

or, taking logarithms,

$$r + \log_{10} k \le n \log_{10} 2 < r + \log_{10}(k+1)$$

Taking fractional parts, this may be written as

$$\log_{10} k \leq \{n\lambda\} < \log_{10}(k+1),$$

where $\lambda = \log_{10} 2$. Since λ is irrational, the sequence $\{n\lambda\}$ is uniformly distributed in $[0,1)$. Therefore, by applying the definition to the interval $[a,b) = [\log_{10} k, \log_{10}(k+1))$, we have that

$$\lim_{n\to\infty} \frac{\xi_n(a,b)}{n} = \log_{10}\left(1+\frac{1}{k}\right).$$

9 Complex Analysis

This section requires a minimum of familiarity with complex power series. The treatment in Spivak [7] suffices. Familiarity with complex analysis is helpful; a standard reference is Ahlfors [2].

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex power series with radius of convergence $R > 0$. For each $0 < r < R$, let f_r be its restriction to the circle $|z| = r$, that is

$$f_r : t \in [0, 2\pi] \mapsto f_r(t) = f(re^{it}).$$

Each f_r can be represented by its Fourier series, which is obtained directly from the power series expression:

$$f_t(t) = \sum_{n=0} a_n r^n e^{int}.$$

that is, $\widehat{f_r}(n) = a_n r^n$ if $n \geq 0$, $\widehat{f_r}(n) = 0$ if $n < 0$. If we use the formula for the Fourier coefficients,

$$\widehat{f_r}(n) = \frac{1}{2\pi}\int_0^{2\pi} f_r(t) e^{-int}\, dt$$

Let $M(r) = \sup_{|z|=r} |f(z)|$, we have

$$|\widehat{f_r}(n)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f_r(t)|\, dt \leq M(r).$$

Proposition 11 *If the power series $f(z) = \sum_{n=0} a_n z^n$ has radius of convergence $R = \infty$, and*

$$\lim_{r\to\infty} \frac{M(r)}{r^k} = 0,$$

then f is a polynomial of degree $\leq k$.

Proof. Clearly, $|a_n| \leq M(r)/r^n$. Thus $a_n = 0$ if $n > k$. □

This result is a generalization of Liouville's theorem of complex analysis (see Ahlfors [2]), which has the fundamental theorem of algebra as a consequence. Another important fact of complex analysis is Cauchy integral formula which we now derive for complex power series.

Consider radii $r_1 < r_2$. Then

$$f(r_1 e^{it}) = \sum_{n=0} a_n r_1^n e^{int}$$

The relation between the Fourier coefficients of f_{r_1} and f_{r_2} says that

$$a_n r_2^n = \frac{1}{2\pi} \int_0^{2\pi} f(r_2 e^{is}) e^{-ins}\, ds,$$

so that we can write

$$\begin{aligned} f(r_1 e^{it}) &= \frac{1}{2\pi} \sum_{n=0} \left(\frac{r_1}{r_2} e^{it} \right)^n \int_0^{2\pi} f(r_2 e^{is}) e^{-ins}\, ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \sum_{n=0} \left(\frac{r_1}{r_2} e^{i(t-s)} \right)^n \right\} f(r_2 e^{is})\, ds. \end{aligned}$$

The series inside the integral is a geometric series whose sum is

$$\sum_{n=0} \left(\frac{r_1}{r_2} e^{i(t-s)} \right)^n = \left(1 - \frac{r_1 e^{it}}{r_2 e^{is}} \right)^{-1},$$

so that

$$f(r_1 e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f(r_2 e^{is}) \left(1 - \frac{r_1 e^{it}}{r_2 e^{is}} \right)^{-1} ds.$$

If we now let $z = r_1 e^{it}$, $w = r_2 e^{is}$, we can rewrite the above expression as

$$f(z) = \frac{1}{2\pi i} \int_{|w|=r_2} \frac{f(w)}{w - z}\, dw,$$

which is know as the Cauchy integral formula in complex analysis.

10 Exercises

Exercise 1. Prove **Parseval's identity**: if $|f|^2$ is integrable, then

$$\sum_{-\infty}^{\infty} |\widehat{f}(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2\, dx.$$

See also Problem 15-13.

Exercise 2. For each positive integer N, let

$$F_N(t) = \frac{1}{N+1} \sum_{n=0}^{N} D_N(t)$$

denote the **Fejer kernel**, defined on $[-\pi, \pi]$.

(a) Prove that

$$F_N(t) = \frac{1}{N+1} \frac{\sin^2((N+1)t/2)}{\sin^2(t/2)}.$$

(b) The function F_N is periodic and non-negative.

(c) The integral $\int_{-\pi}^{\pi} F_N(t)\, dt = 2\pi$.

(d) For each $a > 0$,

$$\lim_{N\to\infty} \int_{a\leq|t|\leq\pi} F_N(t)\, dt = 0.$$

Exercise 3. Let f be integrable on $[-\pi, \pi]$. For each positive integer N, define

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_N f(x).$$

(a) Prove that

$$\sigma_N f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x-t) f(t)\, dt.$$

(b) Show that if f is continuous on $[-\pi, \pi]$, then $\sigma_N f$ converges to f uniformly.

Exercise 4. If f has continuous derivative on $[0, 2\pi]$ then there is a constant M such that $|\widehat{f}(n)| \leq M/|n|$ for all $n \neq 0$.

Exercise 5. Let f be a continuous function on $[0, 2\pi]$, and suppose that $\widehat{f}(n) = 0$ for all n. Show that $f \equiv 0$.

Exercise 6. In physics you probably have told about the Dirac delta function. It is the function δ on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} f(x)\delta(x-a)\, dx = f(a).$$

Although δ is not really a function, you can still compute its Fourier series. What is it?

Can you compute the Fourier series of the derivative δ' of the delta function?

Exercise 7. Compute the Fourier series of the function $f(x) = |x - \pi|$, $0 \leq x \leq 2\pi$.

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