Math 623. Homework 4. Due 03/24/04

Problem 1. Construct the inverse of a square circumscribed about the circle of inversion.

Problem 2. Prove the analogue of Pythagoras' theorem in spherical geometry

$$\cos a = \cos b \cos c$$

where a is the hypotenuse and b, c are the two other sides of a right spherical triangle in the unit sphere.

Problem 3. Let ρ, ρ' be rotations of the sphere. Under what circumstances does $\rho\rho' = \rho'\rho$?

Problem 4. Prove that any two non-intersecting circles can be inverted into concentric circles.

Problem 5. A linear fractional transformation is a map of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0, \qquad a, b, c, d \in \mathbf{C}$$

Let $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. A linear fractional transformation determines a bijection of $\widehat{\mathbf{C}}$. Let $z_1, z_2, z_3, z_4 \in \widehat{\mathbf{C}}$ with z_2, z_3, z_4 distinct.

(a) Find a linear fractional transformation f which carries z_2 to 1, z_3 to 0 and z_4 to ∞.

(b) Define the *cross-ratio* $[z_1, z_2, z_3, z_4]$ as $f(z_1) \in \widehat{\mathbf{C}}$. Show that the cross-ratio is invariant under linear fractional transformations; that is, if g is a linear fractional transformation, then

$$[g(z_1), g(z_2), g(z_3), g(z_4)] = [z_1, z_2, z_3, z_4]$$