Math 623. Homework 3. Due 03/10/04

Problem 1. For each of the following of the following metric spaces (assuming the obvious distance function) state, with a brief justification, whether it is a Euclidean surface, connected or complete.

(a) The union of two parallel planes in $\mathbf{R}^3$.

(b) The surface of an infinite triangular prism.

(c) The surface of a cube.

(d) The surface of a cube minus the vertices.

(e) A close disk.

(f) An open disk.

(g) $\mathbf{C} \setminus \{1/n \mid n = \pm 1, \pm 2, \cdots\}$.

Problem 2. Let Γ be a discontinuous and fixed point free group of isometries of the plane. A fundamental domain for Γ is a region D of the plane that contains one point from each orbit of Γ, and at most one point of each orbit in its interior (that is, if two distinct points of the same orbit are in D, they necessarily lie on the boundary of D). Let D be a fundamental domain for Γ. Prove that

(a) the collection of all the regions gD, $g \in \Gamma$, covers the whole plane,

(b) if two distinct regions intersect they do so only along boundary points, and

(c) given any two such regions there is exactly one isometry in Γ which takes one into the other.

Problem 3. Let Γ be a discontinuous and fixed point free group of isometries of the plane. Let p be a fixed point in the plane and define D to be the set of all points which are closer to p than to any other point of its orbit. That is

$$D = \{z \in \mathbf{C} \mid d(z, p) \leq d(z, gp) \text{ for all } g \in \Gamma\}.$$

(a) Prove that D is a fundamental domain for Γ.

(b) Sketch such construction when Γ is the group generated by $g(z) = z + 1$ and $h(z) = z + 1 + i$.

Problem 4. Let Γ be a discontinuous, fixed point free group of glide reflections and translations. Let g be a glide reflection of minimal length in Γ, and let h be an element of minimal length in Γ not in the direction of g. Prove that g and h must have perpendicular directions (e.g., by finding shorter elements when the directions of g and h are not perpendicular).

Problem 5. Let Γ, g, and h be as in Problem 4. Prove that g and h generate Γ.