Math 623. Homework 2. Solutions

Problem 1. Prove that any isometry of $\mathbb{C}$ has exactly one of the following properties:

(a) A line of fixed points.

(b) A single fixed point.

(c) No fixed point, and a parallel family of invariant lines. (An invariant line is a line mapped onto itself by the isometry.)

(d) No fixed points, and a single invariant line.

Moreover, prove that an isometry f has the property listed above respectively when

(a) $f = r_l$, reflection in the line l.

(b) $f = r_l r_m$, where the lines l, m intersect.

(c) $f = r_l r_m$, where l, m are parallel.

(d) $f = r_l r_m r_n$, where l, m, n have no common point and are not parallel.

Proof. This is a review of the classification of isometries of the Euclidean plane. □

Problem 2. Prove that there is closed line ℓ of minimum length in the twisted cylinder, and that any other closed line has twice the length of ℓ.

(a) What happens to the twisted cylinder if it is cut along this line of minimum length? (b) what happens if it is cut along any other closed line?

Problem 3. Let $T = \mathbf{C}/\Gamma$ be a torus, where Γ is generated by two translations along non-parallel directions.

(a) Prove that any translation of $\mathbf{C}$ induces an isometry of T.

(b) Prove that if the fundamental region for T is a square, then T has an isometry of order 4 (that is, there is a mapping $f : T \to T$ that preserved distances and such that $f^4 = \text{id}$.

(c) Prove that if the fundamental region for T is a rhombus with angles $\pi/3$ and $2\pi/3$, then T has an isometry of order 6 induced by a rotation of $\mathbf{C}$.

Proof. Let τ_1 and τ_2 be the translations that generate Γ. A point p in $\mathbf{C}/\Gamma$ is an orbit of a point P in $\mathbf{C}$ under Γ, that is $p = \{\tau_1^n \tau_2^m P \mid n, m \in \mathbf{Z}\}$. If τ is a translation of the plane, then $\tau\tau_i = \tau_i\tau$, for $i = 1, 2$, because translations of the plane commute. Therefore, τ applied ot the orbit of P is precisely the orbit of τP. That is is, τ sends orbits of Γ to orbits of Γ, ans so induced a map $\widetilde{\tau}$ of the orbit space $\mathbf{C}/\Gamma$ into itself. This map $\widetilde{\tau}$ is a bijection because it has an inverse, namely, the map induced by the inverse translation τ^{-1}. It is also easy to see that it preserves lines and lengths because it is locally an isometry: if $\pi : \mathbf{C} \to \mathbf{C}/\Gamma$ is the orbit map, then $\widetilde{\tau}$ is locally the "composition" $\pi \circ \tau \circ \pi^{-1}$. □

Problem 4. The topic of this problems goes back to Gauss. The problem involves points of the square lattice in the plane. To draw it, you mark out the four corners of a unit square. Then move the square one unit of length in a direction parallel to one of its sides and mark the position of the two new corners, and so on. Gauss tried to determine the number $f(r)$ of lattice points in the interior and on the boundary of a circle of radius r, where the center of the circle is a lattice point and r is an integer. Gauss' interest in this function $f(r)$ is that it yields a method for approximating the value of π, as you will now prove.

(a) Show that $f(r)$ is the area of the region covered by all the unit squares whose lower left-hand corners are inside or on the boundary of the circle. Deduce that the difference between $f(r)$ and the area πr^2 of the circle is at most equal to the combined area $A(r)$ of those squares (the counted as well as the omitted ones) which are cut by the boundary of the circle, that is

$$|f(r) - \pi r^2| \leq A(r).$$

(b) To estimate the area $A(r)$, let $B(r)$ be the area of the annulus of width $2\sqrt{2}$ bounded by the circles of radius $r - \sqrt{2}$ and $r + \sqrt{2}$ centered at the origin, and prove that $A(r) < B(r)$.

(c) Deduce from (a) and (b) that $\lim_{r\to\infty} \frac{f(r)}{r^2} = \pi$.

It is not hard to compute by hand a few values $f(r)$ for $r = 1, 2, 3, \cdots$

Proof. (a) No two different unit squares can share the same lower left-hand corner. Since the squares have unit area, it is the clear that $f(r)$ is the same as the number of the squares mentioned in (a).

The squares contributing to $f(r)$ together with those contributing to $A(r)$ cover a region containing the circle of radius r. Therefore, $f(r) + A(r) > \pi r^2$, or $f(r) - \pi r^2 > -A(r)$. Furthermore, πr^2 is greater than the total area of those unit squares lying inside the circle of radius r, and these squares together with those contributing to $A(r)$ cover a region larger that the one covered by those squares contributing to $f(r)$. That is to say $f(r) < A(r) + \pi r^2$. It follows from these two inequalities that $|f(r) - \pi r^2| < A(r)$.

(b) Any two points in a unit square are at distance $\leq \sqrt{2}$ apart. Thus if a square meets the circle of radius r and center the origin, it cannot have a point lying out side the annulus bounded by the circles of radius $r \pm \sqrt{2}$ and center the origin because any point outside this annulus is at distance $> \sqrt{2}$ from the circle or radius r. It follows that the unit squares contributing to $A(r)$ must lie inside that annulus, and so $A(r) < B(r)$.

(c) The area $B(r) = \pi(r + \sqrt{2})^2 - \pi(r - \sqrt{2})^2 = 4\pi\sqrt{2}r$. Thus $|f(r)/r^2 - \pi| < 4\pi\sqrt{2}/r$, and so $\lim_{r\to\infty} \frac{f(r)}{r^2} = \pi$.

Here are a few values of $f(r)$: $f(10) = 317$, $f(100) = 31417$.

□

Problem 5. Let a be an irrational number. Then, as shown in class, the fractional parts of the numbers na, n an integer, are dense in $[0, 1)$.

(a) Prove that given any positive integer N, there are at least two integers m, n with $1 \leq n \leq N+1$, such that the fractional parts of na and ma are at distance less than $1/N$. In particular, if $r = n - m$ prove that there is an integer k such that

$$\left| a - \frac{k}{r} \right| < \frac{1}{Nr}.$$

(b) Use (a) to prove that given an irrational number a then there are infinitely many rational numbers p/q such that

$$\left| a - \frac{p}{q} \right| < \frac{1}{q^2}.$$

(c) The result in (b) shows that irrational numbers have very good approximations by rational ones. Does the standard method of approximating an irrational number by truncating its decimal expansion have the property of the approximation in (b)?

Proof. This is a theorem of Dirichlet, where the "pigeonhole principle is introduced in part (a). The proof that I wrote for part (b) uses a similar principle. For a real number x, let $\langle x \rangle$ denote its fractional part, and let $[x]$ denote its integer part, so that $x = [x] + \langle x \rangle$ with $\langle x \rangle$ in the interval $[0, 1)$.

(a) Given $N \geq 1$, partition the interval $[0, 1)$ into N equal subintervals of the form $[k/N, (k+1)/N)$, $k = 0, 1, \cdots, N - 1$. Because a is irrational, the fractional parts $\langle na \rangle$, $1 \leq n \leq N + 1$, are all distinct, and thus at least two of then are in the same interval. That is, there are integers n, m (with $1 \leq n < m \leq N + 1$) such that $|\langle na \rangle - \langle ma \rangle| < 1/N$. Thus, if $r = n - m$, then $ra = k + b$ with $0 < b < 1/N$, and so $|ra - k| < 1/N$. Since $1 \leq r < N$, this implies that $|a - k/r| < 1/r^2$.

(b) For an integer $N \geq 1$, let $\mathcal{S}_N$ be the set of rational numbers p/q in reduced form with $q \geq 1$ such that $|a - p/q| < 1/qN$. To proof (b) it suffices to prove that $\mathcal{S}_N$ is an infinite set for each N. Part (a) shows that $\mathcal{S}_N$ is never empty. Also, if $N < M$, then $\mathcal{S}_M \subset \mathcal{S}_N$, and the intersection $\bigcap_{N \geq 1} \mathcal{S}_N = \emptyset$ because a is irrational. If $\mathcal{S}_N$ was finite for some N, then this would imply that $\mathcal{S}_M = \emptyset$ for some $M \geq N$.

(c) No. If the decimal expansion of a is $a = [a] + \dfrac{a_1}{10} + \dfrac{a_2}{10^2} + \cdots$, with $0 \leq a_i \leq 9$, then the rational number $\dfrac{k}{10^n} = [a] + \dfrac{a_1}{10} + \dfrac{a_2}{10^2} + \cdots + \dfrac{a_n}{10^n}$ satisfies $|a - k/10^n| < 1/10^n$, but you cannot improve on this.

$\square$