

## Math 623. Homework 2. Due 02/25/04

**Problem 1.** Prove that any isometry of  $\mathbb{C}$  has exactly one of the following properties:

- (a) A line of fixed points.
- (b) A single fixed point.
- (c) No fixed point, and a parallel family of invariant lines. (An invariant line is a line mapped onto itself by the isometry.)
- (d) No fixed points, and a single invariant line.

Moreover, prove that an isometry  $f$  has the property listed above respectively when

- (a)  $f = r_l$ , reflection in the line  $l$ .
- (b)  $f = r_l r_m$ , where the lines  $l, m$  intersect.
- (c)  $f = r_l r_m$ , where  $l, m$  are parallel.
- (d)  $f = r_l r_m r_n$ , where  $l, m, n$  have no common point and are not parallel.

**Problem 2.** Prove that there is closed line  $\ell$  of minimum length in the twisted cylinder, and that any other closed line has twice the length of  $\ell$ .

What happens to the twisted cylinder if it is cut along this line of minimum length? What happens if it is cut along any other closed line?

**Problem 3.** Let  $T = \mathbb{C}/\Gamma$  be a torus, where  $\Gamma$  is generated by two translations along non-parallel directions.

- (a) Prove that any translation of  $\mathbb{C}$  induces an isometry of  $T$ .
- (b) Prove that if the fundamental region for  $T$  is a square, then  $T$  has an isometry of order 4 (that is, there is a mapping  $f : T \rightarrow T$  that preserved distances and such that  $f^4 = \text{id}$ ).
- (c) Prove that if the fundamental region for  $T$  is a rhombus with angles  $\pi/3$  and  $2\pi/3$ , then  $T$  has an isometry of order 6 induced by a rotation of  $\mathbb{C}$ .

**Problem 4.** The topic of this problems goes back to Gauss. The problem involves points of the square lattice in the plane. To draw it, you mark out the four corners of a unit square. Then move the square one unit of length in a direction parallel to one of its sides and mark the position of the two new corners, and so on. Gauss tried to determine the number  $f(r)$  of lattice points in the interior and on the boundary of a circle of radius  $r$ , where the center of the circle is a lattice point and  $r$  is an integer. Gauss' interest in this function  $f(r)$  is that it yields a method for approximating the value of  $\pi$ , as you will now prove.

- (a) Show that  $f(r)$  is the area of the region covered by all the unit squares whose left-hand corners are inside or on the boundary of the circle. Deduce that the difference between  $f(r)$  and the area  $\pi r^2$  of the circle is at most equal to the combined area  $A(r)$  of those squares (the counted as well as the omitted ones) which are cut by the boundary of the circle, that is

$$|f(r) - \pi r^2| \leq A(r).$$

- (b) To estimate the area  $A(r)$ , let  $B(r)$  be the area of the annulus of width  $2\sqrt{2}$  bounded by the circles of radius  $r - \sqrt{2}$  and  $r + \sqrt{2}$  centered at the origin, and prove that  $A(r) < B(r)$ .
- (c) Deduce from (a) and (b) that  $\lim_{r \rightarrow \infty} \frac{f(r)}{r^2} = \pi$ .

It is not hard to compute by hand a few values  $f(r)$  for  $r = 1, 2, 3, \dots$

**Problem 5.** Let  $a$  be an irrational number. Then, as shown in class, the fractional parts of the numbers  $na$ ,  $n$  an integer, are dense in  $[0, 1)$ .

- (a) Prove that given any positive integer  $N$ , there are at least two integers  $m, n$  with  $1 \leq n \leq N + 1$ , such that the fractional parts of  $na$  and  $ma$  are at distance less than  $1/N$ . In particular, if  $r = n - m$  prove that there is an integer  $k$  such that

$$\left| a - \frac{k}{r} \right| < \frac{1}{Nr}.$$

- (b) Use (a) to prove that given an irrational number  $a$  then there are infinitely many rational numbers  $p/q$  such that

$$\left| a - \frac{p}{q} \right| < \frac{1}{q^2}.$$

- (c) The result in (b) shows that irrational numbers have very good approximations by rational ones. Does the standard method of approximating an irrational number by truncating its decimal expansion have the property of the approximation in (b)?