Math 623. Homework 1. Solutions

Problem 1. Let z_1, z_2 and z_3 be three distinct complex numbers, with $|z_1| = |z_2| = |z_3|$. Prove that the following properties are equivalent:

(a) The points are equidistant.

(b) $z_1 + z_2 + z_3 = 0$.

(c) There is a number a such that z_1, z_2, z_3 are the roots of the equation $z^3 = a$.

Proof. Let z, w be two complex numbers such that $|z| = |w| = 1$. If the points $1, z, w$ are equidistant, then $|1 - z| = |1 - w| = |z - w|$, and so $z + \overline{z} = w + \overline{w} = \overline{w}z + w\overline{z}$. Because $\overline{z} = 1/z$ and $\overline{w} = 1/w$, it follows that $z + \dfrac{1}{z} = \dfrac{w}{z} + \dfrac{z}{w}$, or that $w(z^2 + 1) = w^2 + z^2$. Therefore $z^2 = w$, and similarly $w^2 = z$. Thus $z^3 = wz = w^3$. Thus $z^4 = w^2 = z$, so $z^3 = w^3 = 1$. From the above $1 + z + w = 1 + z + z^2 = \dfrac{z^3 - 1}{z - 1} = 0$.

Now we solve the problem. Given z_i, $i = 1, 2, 3$, with $|z_i| = r$, let $z = z_2/z_1$ and $w = z_3/z_1$.
(a) $\Rightarrow$ (b) If the points z_i are equidistant, the the points $1, z, w$ are also equidistant. If follows from the calculations above that $1 + z + w = 0$, hence that $z_1 + z_2 + z_3 = 0$.
(b) $\Rightarrow$ (c) If $z_1 + z_2 + z_3 = 0$, then $1 + z + w = 0$. Multiply by z to get $z + z^2 + zw = 0$. Take the conjugate to get $1 + \dfrac{1}{z} + \dfrac{1}{w} = 0$, so $wz + w + z = 0$. It follows that $z^2 = w$. Thus $0 = 1 + z + w = 1 + z + z^2 = \dfrac{1 - z^3}{1 - z}$, which implies $z^3 = 1$. Similarly $w^3 = 1$. It follows that $z_1^3 = z_2^3 = z_3^3 = a$ with $a = z_1^3$.
(c) $\Rightarrow$ (a) The three roots of $z^3 = a$ are the vertices of an equilateral triangle. Algebraically, if z_i, $i = 1, 2, 3$ are the three roots of $z^3 = a$, then the numbers $1, z, w$ are the three roots of unity. Then $\overline{z} = w$ and $\overline{w} = z$, $z^2 = w$ and $w^2 = z$. It follows that $|z - w| = |z||1 - w/z| = |1 - z|$ and similarly $|z - w| = |1 - w|$. This implies that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$. ☐

Problem 2. Let s_n denote the side length of P_n, the regular n-sided polygon inscribed in the unit circle.

(a) Prove that

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

Deduce from this that

$$s_4 = \sqrt{2}, \qquad s_8 = \sqrt{2 - \sqrt{2}}, \qquad \cdots \qquad s_{2^{n+1}} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}},$$

(in the last expression there are n nested square roots).

(b) Prove that

$$\lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}} = \pi$$

(in the limit there are n nested square roots).

Proof. (a) follows easily by induction using elementary trigonometry.
(b) A regular m-gon P_m inscribed in the unit circle circumscribes a disk of radius $\sin(\pi/m)$. Therefore, the area of P_m satisfies $\pi \sin^2(\pi/m) < \mathrm{Area}(P_m) < \pi$, hence $\lim_{m\to\infty} \mathrm{Area}(P_m) = \pi$. Now use (a) with $m = 2^n$. ☐

Problem 3. (a) Prove that a line in $\mathbf{C}$ satisfies an equation of the form

$$\overline{a}z + a\overline{z} + b = 0$$

(a complex and b real), and conversely.

(b) Prove that a line in $\mathbf{C}$ can be described as the set of points in $\mathbf{C}$ equidistant from two fixed points, and use this fact to show that isometries take lines to lines.

(c) Prove that if ℓ is the line equidistant from the points a and b, then reflection on ℓ takes a to b.

Proof. (a) In Cartesian coordinates (x, y), a line is given by a linear equation of the form $2Ax+2By+C = 0$, for some real numbers A, B, C. Use that $2x = z + \overline{z}$ and $2iy = z - \overline{z}$ to write that equation in the form $A(z + \overline{z}) - iB(z - \overline{z}) + C = 0$. This simplifies to $(A - iB)z + (A + iB)\overline{z} + c = 0$, as desired. ☐

Problem 4. (a) Prove that the map

$$z \mapsto e^{i\alpha}z + v$$

is a translation if $e^{i\alpha} = 1$ and otherwise a rotation about $v/(1 - e^{i\alpha})$.

(b) Prove that

$$z \mapsto e^{i\alpha}\overline{z} + v$$

is a glide reflection. Find its axis and the length of translation.

Proof. (a) The map $f(z) = e^{i\alpha}z + v$ is an isometry because $|f(z) - f(w)| = |e^{i\alpha}z - e^{i\alpha}w| = |z - w|$. If $e^{i\alpha} = 1$, then f is a translation. If $e^{i\alpha} \neq 1$, then $f(1/1 - e^{i\alpha}) = 1/1 - e^{i\alpha}$. Thus f has exactly one fixed point, and so it must be a rotation.
(b) The map $g(z) = e^{i\alpha}\overline{z} + v$ is the composite $g = f \circ r$, where $r(z) = \overline{z}$. Because of (a) and the classification of isometries, f is the product of two reflections. Thus g is the product of three reflections, and so it must be a glide reflection. It could be a trivial glide reflection, that is, just a reflection.

The map g^2 is a translation (it could be the identity if g was just a reflection) whose direction is that of the axis of g and whose length is twice the length of g. Since $g^2(z) = z + e^{i\alpha}\overline{v} + v$, the length of g is $|e^{i\alpha}\overline{v} + v|/2$. The axis is the line that passes through $v/2$ and has direction vector $e^{i\alpha}\overline{v} + v$ (it passes through $v/2$ because the axis bisects the line segment joining 0 and $g(0)$). ☐

Problem 5. (a) Prove that two rotations by the same angle ρ_1 and ρ_2 are conjugate by an isometry, that is, there is an isometry f such that $\rho_2 = f \circ \rho_1 \circ f^{-1}$.

(b) When are two translations conjugate?

Proof. (a) If P and Q are the centers of rotation of ρ_1 and ρ_2, let τ be the translation by $P - Q$.
(b) Let $\tau_1(z) = z + u_1$ and $\tau_2(z) = z + u_2$. They are conjugate by $f(z) = e^{i\alpha}z + v$, that is, $\tau_2 = f\tau_1 f^{-1}$, if and only if $u_2 = e^{i\alpha}u_1$. They are conjugate by $g(z) = e^{i\alpha}\overline{z} + v$ if and only if $u_1 = u_2$. ☐