Math 623. Homework 1. Due 02/11/04

Problem 1. Let z_1, z_2 and z_3 be three distinct complex numbers, with $|z_1| = |z_2| = |z_3|$. Prove that the following properties are equivalent:

(a) The points are equidistant.

(b) $z_1 + z_2 + z_3 = 0$.

(c) There is a number a such that z_1, z_2, z_3 are the roots of the equation $z^3 = a$.

Problem 2. Let s_n denote the side length of P_n, the regular n-sided polygon inscribed in the unit circle.

(a) Prove that

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

Deduce from this that

$$s_4 = \sqrt{2}, \qquad s_8 = \sqrt{2 - \sqrt{2}}, \quad \cdots \qquad s_{2^{n+1}} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}},$$

(in the last expression there are n nested square roots).

(b) Prove that

$$\lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}} = \pi$$

(in the limit there are n nested square roots).

Problem 3. (a) Prove that a line in $\mathbf{C}$ satisfies an equation of the form

$$\bar{a}z + a\bar{z} + b = 0$$

(a complex and b real), and conversely.

(b) Prove that a line in $\mathbf{C}$ can be described as the set of points in $\mathbf{C}$ equidistant from two fixed points, and use this fact to show that isometries take lines to lines.

(c) Prove that if ℓ is the line equidistant from the points a and b, then reflection on ℓ takes a to b.

Problem 4. (a) Prove that the map

$$z \mapsto e^{i\alpha} z + v$$

is a translation if $e^{i\alpha} = 1$ and otherwise a rotation about $v/(1 - e^{i\alpha})$.

(b) Prove that

$$z \mapsto e^{i\alpha}\overline{z} + v$$

is a glide reflection. Find its axis and the length of translation.

Problem 5. (a) Prove that two rotations by the same angle ρ_1 and ρ_2 are conjugate by an isometry, that is, there is an isometry f such that $\rho_2 = f \circ \rho_1 \circ f^{-1}$.

(b) When are two translations conjugate?