

Problems for Solution

Problem 1. Let $\gamma: [a, b] \rightarrow U$ be a piecewise smooth path in U and let ω be a one form on U .

(i) Define: *integral of ω along γ* .

(ii) Let γ be the path constructed as follows: the line segment from the point $(1, 0)$ to the point $(0, 1)$, followed by the top arc of the circle of center $(1/2, 1)$ and radius $1/\sqrt{2}$ from $(0, 1)$ to $(1, 1)$, and followed by the line segment from $(1, 1)$ to the origin. Evaluate the path integral $\int_{\gamma} \omega$, where ω is the 1-form $\omega = -ydx + xdy$.

Problem 2. Given a 1-form ω on an open set U , prove that the following are equivalent (i) $d\omega = 0$; (ii) $\int_{\partial R} \omega = 0$ for all closed rectangles R contained in U ; (iii) every point in U has a neighborhood such that $\int_{\partial R} \omega = 0$ for all closed rectangles contained in the neighborhood. Is the same true if closed rectangles are replaced by disks?

Problem 3. (i) Define the following concepts: (a) *differential 1-form*, (b) *closed 1-form*, and (c) *exact 1-form*.

(ii) Prove that any closed 1-form on \mathbf{R}^2 is exact.

Problem 4. Let $\gamma: [a, b] \rightarrow \mathbf{R}^2 \setminus \{P\}$ be a continuous path.

(i) Define the concept: *winding number $W(\gamma, P)$ of γ about P* .

(ii) Let \mathbf{v} be a vector in the plane and let $\gamma + \mathbf{v}$ denote the path in $\mathbf{R}^2 \setminus \{P + \mathbf{v}\}$ defined by $(\gamma + \mathbf{v})(t) = \gamma(t) + \mathbf{v}$. Prove that

$$W(\gamma + \mathbf{v}, P + \mathbf{v}) = W(\gamma, P).$$

Problem 5. Let $\gamma, \delta: [0, 2\pi] \rightarrow U$ be the paths defined by $\gamma(t) = (2\cos 2t, 2\sin 2t)$ and $\delta(t) = (2\cos 2t + \cos t, 2\sin 2t + \sin t)$. Prove that

$$W(\gamma, 0) = W(\delta, 0).$$

Problem 6. Define the operator $*$ (called ‘star’) on j -forms as follows: for a function (0-form) f , let $*f = f dx dy$; for a 1-form $pdx + qdy$, let $*(pdx + qdy) = -qdx + pdy$; and for a 2-form $h dx dy$, let $*(h dx dy) = h$. Thus, the ‘star’ operator takes j -forms to $(2 - j)$ -forms, $j = 0, 1, 2$.

The Laplacian of a function f on U is the function $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. A function f such that $\Delta f = 0$ is called *harmonic*.

(i) Prove that if f is a smooth function on U , then $*d*df = \Delta f$.

(ii) Prove that a 1-form $\omega = pdx + qdy$ is such that ω and $*\omega$ are both closed if and only if the functions p and q are harmonic.

Problem 7. (i) Define *homotopic relative to endpoints* and *homotopic as closed paths* for continuous paths in an open sets U .

(ii) Prove that if U is an open convex set, then any two continuous paths from $[a, b]$ to U with the same endpoints are homotopic relative to their endpoints in U , and any two closed paths in U are homotopic as closed paths in U .

Problem 8. Prove that if ω is a closed 1-form on $\mathbf{R}^2 \setminus \{0\}$ such that $\int_{\gamma} \omega = 0$, where γ is $\gamma(t) = (\sin t, \cos t)$ ($0 \leq t \leq 2\pi$), then ω is exact on $\mathbf{R}^2 \setminus \{0\}$.

(Make sure that you state carefully all the results that you are using in your proof.)

Problem 9. For any point $P = (x_0, y_0)$, let $\omega_{P,\theta}$ be the 1-form on $\mathbf{R}^2 \setminus \{P\}$ defined by

$$\omega_{P,\theta} = \frac{1}{(x-x_0)^2 + (y-y_0)^2} \left(-(y-y_0)dx + (x-x_0)dy \right).$$

Prove that for any two points P and Q , the 1-form $\omega = \omega_{P,\theta} - \omega_{Q,\theta}$ is exact on $\mathbf{R}^2 \setminus L$, where L is the line segment from P to Q .