Math 550. Homework 9. Due 12/03/03

Problem 1. Suppose that $U = \mathbf{R}^2 \setminus \{P_1, \cdots, P_n\}$ is the complement of n points in the plane. Prove that the mapping that takes a closed 1-chain γ to $\big(W(\gamma, P_1), \cdots, W(\gamma, P_n)\big)$ determines an isomorphism of $H_1 U$ with the free abelian group $\mathbf{Z}^n$.

Problem 2. (i) Prove that a continuous mapping $F : X \to Y$ determines a homomorphism from $Z_1 X$ to $Z_1 Y$ taking $B_1 X \to B_1 Y$, and thus it determines a homomorphism of abelian groups $F_* : H_1 X \to H_1 Y$.

(ii) Prove that if $F : X \to Y$ and $G : Y \to Z$ are continuous, then $(G \circ F)_* = G_* \circ F_*$ as homomorphism from $H_1 X$ to $H_1 Z$. In particular, prove that if X and Y are homeomorphic space, then $H_1 X$ and $H_1 Y$ are isomorphic abelian groups.

Problem 3. Find examples of continuous mappings $F : X \to Y$ such that:

(i) F is one-one, but F_* is not one-one.

(ii) F is surjective, but F_* is not surjective.

Problem 4. Let K be a compact subset of the plane and let $U = \mathbf{R}^2 \setminus K$. Prove that if K is not connected, and P and Q are in different components of K, then the classes $[\omega_P]$ and $[\omega_Q]$ are linearly independent in $H^1 U$. In particular, prove that if K has n connected components, then $H^1 U$ is a vector space of dimension n.

Problem 5. Compute the integral $\displaystyle\int_\gamma \omega$, where ω is the 1-form

$$\omega = \sum_{n=1}^{17} \frac{1}{(x-n)^2 + y^2} \big(-y\,dx + (x-n)\,dy\big)\,,$$

and $\gamma(t) = (t\cos(t), t\sin(t))$, $0 \le t \le 6\pi$.