

Math 550. Homework 8. Solutions

Problem 1. Let U, V be open subsets of \mathbf{R}^2 . Prove that the coboundary map $\delta : H^0(U \cap V) \rightarrow H^1(U \cup V)$ is a homomorphism of vector spaces.

Solution. This is in the textbook. □

Problem 2. Let $F : U \rightarrow V$ be a smooth map from the open set $U \subset \mathbf{R}^2$ into the open set $V \subset \mathbf{R}^2$. In a previous homework we showed that F induces a “pull-back” operator F^* that sends n -forms on V into n -forms on U . Prove that F induces a linear map of vector spaces $F^* : H^n V \rightarrow H^n U$, for $n = 0, 1$.

Moreover, prove that if F is a diffeomorphism, then $F^* : H^n V \rightarrow H^n U$ is an isomorphism of vector spaces.

Solution. The pull-back F^* is linear on 1-forms and 0-forms. That is, if ω_1 and ω_2 are two 1-forms (or two 0-forms), then $F^*(\omega_1 + \omega_2) = F^*(\omega_1) + F^*(\omega_2)$, and if λ is a scalar, then $F^*(\lambda\omega_1) = \lambda F^*(\omega_1)$. This was done in homework 4 (look carefully at the solution).

I will show that F^* induces a linear map (also called F^*) from $H^1 V$ to $H^1 U$. (The case when $n = 0$ is exactly the same.) Since H^1 is closed 1-forms modulo exact 1-forms, this amounts to showing that if ω is a closed 1-form on V , then $F^*\omega$ is a closed 1-form on U , and that if ω is an exact 1-form on V , then $F^*(\omega)$ is a closed 1-form on U .

We showed in Homework 4 that $dF^* = F^*d$. Thus if ω is closed on V , then $d\omega = 0$, and so $dF^*(\omega) = F^*(d\omega) = F^*(0) = 0$, proving that $F^*(\omega)$ is closed on U .

If ω is exact on V , then $\omega = df$ for some function f on V . Then $F^*(\omega) = F^*(df) = dF^*(f) = d(f \circ F)$, so $F^*\omega$ is exact on U .

In general, if $F : U \rightarrow V$ and $G : V \rightarrow W$ are smooth functions, then the pull-backs satisfy $(G \circ F)^* = F^* \circ G^*$. (This is essentially the chain rule.)

Then, if $F : U \rightarrow V$ is a diffeomorphism, then there is a smooth function $G : V \rightarrow U$ such that $F \circ G$ is the identity on U and $G \circ F$ is the identity on V . (That is, $G = F^{-1}$.) But the identity map $I_U : U \rightarrow U$ induces the identity map $I_{H^1 U} : H^1 U \rightarrow H^1 U$, and similarly the identity map $I_V : V \rightarrow V$ induced the identity map $I_{H^1 V} : H^1 V \rightarrow H^1 V$. Therefore,

$$(F \circ F^{-1})^* = (F^{-1})^* \circ F^* = I_{H^1 V}$$

and

$$(F^{-1} \circ F)^* = (F)^* \circ (F^{-1})^* = I_{H^1 U}$$

From linear algebra you know that this implies that F^* is both one-one and onto, therefore an isomorphism. □

Problem 3. Prove that if U and V are connected, and $H^1(U \cup V) = 0$, then $U \cap V$ is connected.

Solution. Not solution is available at this time. □

Problem 4. Prove that if the open set $U \subset \mathbf{R}^2$ can be written as a union $U = U_1 \cup \dots \cup U_n$, where each U_j is a convex open set, then $H^1 U$ is a finite dimensional vector space.

Solution. All of you will get full credit for this problem because the solution required a fact from linear algebra that may not be very familiar to you. However, the cases $n = 2$ and $n = 3$ can be done without that, using very elementary methods.

Consider the case $n = 2$. If U_1 and U_2 are convex, then $U_1 \cap U_2$ is also convex, hence connected, so $H^0(U_1 \cap U_2)$ has dimension ≤ 1 . Therefore, if we show that the coboundary map $\delta : H^0(U_1 \cap U_2) \rightarrow$

$H^1(U_1 \cup U_2)$ is surjective, then it will follow that $H^1(U_1 \cup U_2)$ is of dimension ≤ 1 . Indeed, if ω is a closed one form on $U_1 \cup U_2$, then it is closed on the convex sets U_1 and U_2 , and thus it is exact on U_1 and U_2 . This says that every class $[\omega]$ in $H^1(U \cup U_2)$ is in the image of δ .

The case when $n = 3$ can be done similarly, but the general case is slightly more difficult than I thought because it uses a result of linear algebra that may not sound too familiar to you. This result is the following. Suppose that A , B , and C are vector spaces and that we have linear maps $f : A \rightarrow B$ and $g : B \rightarrow C$. If A and C are finite dimensional, and the image of f equals the kernel of g , then B is also finite dimensional. Indeed, the dimension of B is the dimension of the kernel of g plus the dimension of the image of g . The image of g is a subspace of C , so it is finite dimensional. The kernel of g is also finite dimensional because it equals the image of f , and this image is finite dimensional because A is finite dimensional.

Using this linear algebra fact we can proceed by induction. Assuming that we have proved that $H^1(U_1 \cup \dots \cup U_{n-1})$ is finite dimensional, we will prove that $H^1(U_1 \cup \dots \cup U_n)$ is also finite dimensional. Consider the coboundary map

$$\delta : H^0((U_1 \cup \dots \cup U_{n-1}) \cap U_n) \rightarrow H^1(U_1 \cup \dots \cup U_n)$$

Now $(U_1 \cup \dots \cup U_{n-1}) \cap U_n = (U_1 \cap U_n) \cup \dots \cup (U_{n-1} \cap U_n)$, and each intersection $U_k \cap U_n$ is convex, hence connected. Therefore $H^0((U_1 \cup \dots \cup U_{n-1}) \cap U_n)$ has dimension at most $n - 1$. (A locally constant function on $(U_1 \cap U_n) \cup \dots \cup (U_{n-1} \cap U_n)$ must be identically constant on the intersection $U_k \cap U_n$ for $k = 1, \dots, n - 1$, so you have at most $n - 1$ constants to play with for constructing such locally constant function.)

Consider the linear mapping

$$g : H^1(U_1 \cup \dots \cup U_n) \rightarrow H^1(U_1 \cup \dots \cup U_{n-1})$$

that to a class $[\omega]$ in $H^1(U_1 \cup \dots \cup U_n)$ assigns the class of the restriction of ω to $U_1 \cup \dots \cup U_{n-1}$. We want to show that the kernel of g equals the image of the coboundary map δ . If $[\omega]$ is in the kernel of g , then ω is exact on $U_1 \cup \dots \cup U_{n-1}$. Since U_n is convex, ω is also exact in U_n . This implies that $[\omega]$ is in the image of δ , as shown in class. Conversely, suppose that $[\omega]$ is in the image of δ . Again, this means that ω is exact on U_n and on $U_1 \cup \dots \cup U_{n-1}$. Hence the class of ω in $H^1(U_1 \cup \dots \cup U_{n-1})$ is 0, or $[\omega]$ is in the kernel of g . \square

Problem 5. Let $U \subset \mathbf{R}^2$ be the complement of n points. Prove that H^1U is a vector space of dimension n , and find a basis for it.

Solution. Suppose that $U = \mathbf{R}^2 \setminus \{P_1, \dots, P_n\}$. First I want to find n parallel lines containing exactly one of the points P_j . To do this, say that a direction is bad if there is a line in that direction containing at least two of the points P_1, \dots, P_n , and good otherwise. Then notice that if a line contains two (or more) points P_j and P_k , then that line is in the direction of the vector $P_j - P_k$. Therefore there are only finitely many bad directions, namely those given by the vectors $P_j - P_k$, $j \neq k$. Since the set of directions of lines in the plane is infinite (it can be parametrized by the interval $[0, \pi)$, there are many directions that are good (i.e., not bad). A line that is parallel to a good direction can contain at most one of the n points P_j . In conclusion, there are n parallel lines ℓ_1, \dots, ℓ_n such that P_j is in ℓ_j for $j = 1, \dots, n$.

Assume that these lines are horizontal. (This is no loss of generality because you can always make them horizontal by a rotation of the plane; a rotation is a diffeomorphism, and you can then apply Problem 2.) For each line ℓ_j , let R_j be the closed ray in ℓ_j to the right of P_j and let L_j be the ray to the left of P_j . Let U_L be the open set $\mathbf{R}^2 \setminus (L_1 \cup \dots \cup L_n)$ and let U_R be $\mathbf{R}^2 \setminus (R_1 \cup \dots \cup R_n)$. Then $U = U_R \cup U_L$ and $U_R \cap U_L = \mathbf{R}^2 \setminus (\ell_1 \cup \dots \cup \ell_n)$.

Consider the coboundary map

$$\delta : H^0(U_R \cap U_L) \rightarrow H^1(U_R \cup U_L) = H^1U.$$

The intersection $U_R \cap U_L$ is the complement of n parallel lines and it thus has $n + 1$ connected components, so $\dim H^0(U_R \cap U_L) = n + 1$. Since the U_R and U_L are both connected, the kernel of δ consists of the constants functions on $U_R \cap U_L$, and so it has dimension 1. By the rank-nullity theorem says

$$\dim H^0(U_R \cap U_L) = (n + 1) = \text{rank } \delta + \text{null } \delta = \text{rank } \delta + 1,$$

so if we prove that δ is onto we will obtain

$$\dim H^1(U) = \text{rank } \delta = n.$$

As proved in class, the image of δ consists of the classes $[\omega]$, where ω is a closed 1-form on U that is exact on U_R and on U_L . Therefore, in order to show that δ is onto it suffices to show that if ω is a closed one form on U_R then it is exact on U_R (and the same for U_L , but note that U_R and U_L have the same shape).

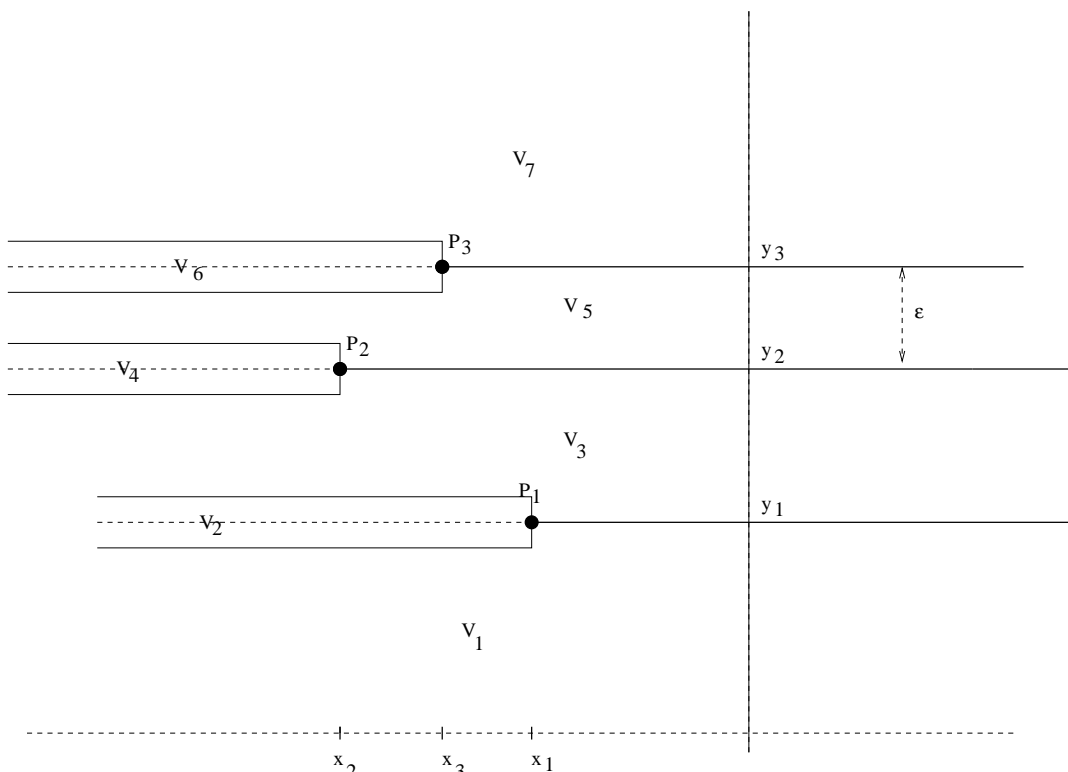
Now U_R can be written as a union $U_R = V_1 \cup V_2 \cup \dots \cup V_{2n+1}$, where each V_k is an open rectangle and where the intersection $(V_1 \cup \dots \cup V_k) \cap V_{k+1}$ is also an open rectangle. This is easy to do in several ways. One such way was mentioned in class and another is sketched in the figure below. To write it down we do as follows. Suppose that the points $P_j = (x_j, y_j)$, and that they are ordered consecutively starting from the bottom so that $y_1 < y_2 < \dots < y_n$. Let $\epsilon = \min\{|y_i - y_{i+1}| \mid i = 1, 2, \dots, n-1\}$ (that is, the smallest distance between consecutive lines). Set also $y_0 = -\infty$ and $y_{n+1} = \infty$. Then take

$$V_{2k} = (-\infty, x_k) \times (y_k - \epsilon/4, y_k + \epsilon/2), \quad \text{for } k = 1, 2, \dots, n,$$

and

$$V_{2k+1} = (-\infty, \infty) \times (y_k, y_{k+1}), \quad \text{for } k = 0, 1, \dots, n,$$

and finally apply Exercise 1.36 in the textbook.



Once we know that $H^1(\mathbf{R}^2 \setminus \{P_1, \dots, P_n\})$ has dimension n we can easily find a basis for it. There are two ways of doing that.

The first way is as follows. Let $r = (1/3) \min\{|P_i - P_j| \mid i \neq j\}$ and let γ_i be the circle of radius r about P_i . Then the form ω_{P_i} satisfies $\int_{\gamma_i} \omega_j = 1$ if $i = j$, and $\int_{\gamma_i} \omega_j = 0$ if $i \neq j$. It follows from this that the classes $[\omega_1], \dots, [\omega_n]$ are linearly independent in $H^1(\mathbf{R}^2 \setminus \{P_1, \dots, P_n\})$, and therefore they form a basis.

The second way is as follows. Let W_1, W_2, \dots, W_{n+1} be the $n+1$ connected components of the intersection $U_R \cap U_L$ (the complement of n parallel lines). Let f_i be the locally constant function on $U_R \cap U_L$ that is identically 1 on W_i and 0 on W_j for $j \neq i$. The collection f_1, \dots, f_{n+1} form a basis of $H^0(U_R \cap U_L)$, thus their images $\delta(f_1), \dots, \delta(f_{n+1})$ span $H^1(\mathbf{R}^2 \setminus \{P_1, \dots, P_n\})$. I leave to you, as an exercise, to show that, for any $k = 1, \dots, n+1$, the set $\{\delta(f_j) \mid j \neq k\}$ is a basis for $H^1(\mathbf{R}^2 \setminus \{P_1, \dots, P_n\})$.

□