# Math 550. Homework 7. Solutions

**Problem 1.** A vector $X$ in $\mathbf{R}^n$ is called a probability vector if its coordinates are all nonnegative and add up to 1. An $n \times n$ matrix is an stochastic matrix if its columns are probability vectors. Use the Brouwer fixed point theorem to prove that if $A$ is a $3 \times 3$ stochastic matrix then there is a probability vector $X$ such that $AX = X$.

*Solution.* Let $P$ denote the set of probability vectors in $\mathbf{R}^3$. Element of $P$ are vectors $(x_1, x_2, x_3)$ such that $x_1, x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 = 1$. This set is homeomorphic to a closed triangle in the plane. Indeed, the mapping that sends $(x_1, x_2, x_3)$ to $(x_1, x_2)$ is a homeomorphism of $S$ onto the triangle in the plane bounded by the coordinate axes and the line $x + y = 1$.

It is very easy to show that a triangle is homeomorphic to a disk. If $T$ is the triangle in the plane, let $D$ be the inscribed circle. Assume for simplicity that the center of $D$ is the origin in the plane, and define a mapping $f$ from $T$ onto the init disk as follows. If $P$ is in $T$, then let $Q_P$ be the point of intersection of the ray from 0 (the center of $D$) $P$ with the boundary of the triangle $T$, and set

$$f(P) = \frac{P}{|Q_P|}.$$

It is a good exercise to show that $f$ is a homeomorphism of $T$ onto the unit disk.

Because a (closed) disk has the fixed point prperty, the set $P$ of probability vectors also has the fixed point property (see first part of Exercise 4.7, which was done in class). If we show that the result of multiplying an stochastic matrix $A$ by a probability vector results in a probability vector, the the fixed point property of $P$ will imply that there is a probability vectors $X$ such that $AX = X$.

Suppose that $A$ is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where each column is a probability vector, that is $a_{1i} + a_{2i} + a_{3i} = 1$ for $i = 1, 2, 3$, and all entries $a_{ij} \geq 0$. Then the vector $AX$ is

$$AX = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

Each coordinate of this vector $AX$ is non-negative because the coordinates of $X$ are non-negative, and the entries of $A$ are non-negative. Furthermore, the sum of all three coordinates of $AX$ equals 1 (it is much easier to prove this by visually inspecting the expression for $AX$ above than it is to write down). $\square$

**Problem 2.** Let $f, g$ be two continuous mappings from $X$ into the unit circle $S^1$. Prove that if $|f(x) - g(x)| < 2$ for all $x$ in $X$, then $f$ and $g$ are homotopic.

*Solution.* To say that for each $x$ in $X$ the points $f(x)$ and $g(x)$ in the unit circle $S^1$ are at distance $< 2$ is the same as to say that the points $f(x)$ and $g(x)$ are never antipodal. Therefore, the hypothesis of the problem implies that for each $x$ in $X$ the segment from $f(x)$ to $g(x)$ never contains the origin in the plane $\mathbf{R}^2$, that is, $tg(x) + (1-t)f(x) \neq 0$ for every $x$ in $X$ and every $t$ in $[0,1]$.

Therefore, the map $H : X \times [0,1] \to S^1$ given by

$$H(x,t) = \frac{tg(x) + (1-t)f(x)}{|tg(x) + ((1-t)f(x)|}$$

is well defined, continuous and satisfies $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x$ in $X$. That is $H$ is a homotopy from $f$ to $g$. □

**Problem 3.** Prove that if the sphere is covered by three closed subsets, then one of them must contain a pair of antipodal points.

**Note.** This is Proposition 4.33 in the textbook. The proof there is not complete; you have to provide a complete proof.

*Solution.* The only thing missing from the proof in the textbook is the following (Exercise 4.34): If $A$ is a subset of $\mathbf{R}^3$ (or any metric space) then the distance to $A$ function is continuous. The distance to $A$ function is given by

$$P \mapsto d(P,A) = \inf\{d(P,Q) \mid Q \text{ in } A\}.$$

We will prove that

$$|d(P,A) - d(Q,A)| \leq d(P,Q)$$

for all points $P$ and $Q$. Continuity follows from this by the usual $\varepsilon - \delta$ argument.

First, we always have the triangle inequality

$$d(P,R) \leq d(P,Q) + d(Q,R)$$

If you fix $R$ in $A$, then $d(P,A) \leq d(P,R) \leq d(P,Q) + d(P,R)$, because of the definition of $d(P,A)$. That is, the number $d(P,A)$ is a lower bound for the set of numbers $\{d(P,Q) + d(P,R) \mid R \in A\}$, and so it cannot be larger that the infimum (greatest lower bound) of this set of numbers, that is to say

$$d(P,A) \leq \inf\{d(P,Q) + d(P,R) \mid R \text{ in } A\} = d(P,Q) + d(Q,A)$$

Switching the roles of $P$ and $Q$ we also obtain that $d(Q,A) \leq d(P,Q) + d(Q,A)$, and thus that

$$|d(P,A) - d(Q,A)| \leq d(P,Q).$$

□

**Problem 4.** Prove that the sphere can be covered with four closed subsets, neither of them containing a pair of antipodal points.

*Solution.* Inscribe a regular tetrahedron in the unit sphere and orthogonally project it onto the surface of the sphere from the center of the sphere. The result is a “spherical” regular tetrahedron. The four faces of this tetrahedron are the four closed subsets covering the sphere that the problem asks for. Indeed, no face can contain a pair of antipodal points because each face is contained in an open hemisphere. □

**Problem 5.** Suppose that $A$ is a connected closed subset of $\mathbf{R}^2$, and $P$ is a point in $A$. Prove that $[\omega_P] = 0$ in $H^1(\mathbf{R}^2 \setminus A)$ if and only if $A$ is unbounded.

*Solution.* Suppose that $A$ is unbounded. We have to show that the closed 1-form $\omega_P$ is exact on $\mathbf{R}^2 \setminus A$. This is equivalent to showing that $\int_\gamma \omega_P = 0$ for every closed, piecewise smooth path $\gamma$ in $\mathbf{R}^2 \setminus A$.

But $\int_\gamma \omega_P = W(\gamma, P)$, and this winding number is 0 because $P$ is in the unbounded component of the complement of the image of $\gamma$. Indeed, $p$ is in $A$ and $A$ is a connected subset of $\mathbf{R}^2 \setminus A$, so $A$ must be contained in the unbounded component of the complement of the image of $\gamma$

Suppose that $[\omega_P] = 0$ in $H^1(\mathbf{R}^2 \setminus A)$, that is, suppose that $\omega_P$ is exact on $R^2 \setminus A$. This implies that $\int_\gamma \omega_P = 0$ for every closed, piecewise smooth path $\gamma$ in $\mathbf{R}^2 \setminus A$. If $A$ was not unbounded, then there would be a disk with center 0 and radius $r$ containing $A$ in its interior. If $\gamma_r$ is the circle of radius $r$ and center 0, then the winding number $W(\gamma, P) = 1$, that is $\int_{\gamma_r} \omega_P = 1$, contradicting the fact that $\omega_P$ is exact. $\square$