Math 550. Homework 6. Solutions

Definition 1. Two continuous mappings $f,g : X \to Y$ are homotopic if there is a continuous mapping $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all x in X.

Problem 1. Let C and C' be circles.

(i) Prove that if a continuous mapping $F : C \to C'$ is not surjective, then $\deg F = 0$.

(ii) Find an example of a continuous mapping $F : C \to C'$ that is surjective but has $\deg F = 0$.

Solution. (i) If F is not surjective, then there is a point P in C' not contained in $F(C)$. It follows that $F(C)$ is contained in a sector (the complement of the ray issuing from the center of C' and passing through P). As we have shown before, this implies that $\deg F = 0$.
(ii) Assume that $C = C'$ are the unit circle. Use polar coordinates $(\cos t, \sin t)$, $0 \le t \le 2\pi$ and define F by

$$F(\cos t, \sin t) = \begin{cases} (\cos 2t, \sin 2t), & 0 \le t \le \pi \\ (\cos 2t, -\sin 2t), & \pi \le t \le 2\pi. \end{cases}$$

This mapping F wraps the upper semicircle and the lower semicircle each onto the full circle, but in opposite directions.

The degree of F is the winding number of the path $\gamma(t) = F(\cos t, \sin t)$ around the origin. We can write γ as a sum of two paths $\gamma = \gamma_1 + \gamma_2$, where

$$\gamma_1(t) = (\cos 2t, \sin 2t), \quad 0 \le t \le \pi$$

and

$$\gamma_2(t) = (\cos 2t, -\sin 2t), \quad \pi \le t \le 2\pi$$

Then

$$W(\gamma, 0) = W(\gamma_1, 0) + W(\gamma_2, 0) = 1 - 1 = 0.$$

□

Problem 2. (i) Prove that two continuous mappings $F, G : C \to C'$ are homotopic if and only if they have the same degree.

(ii) Conclude that a continuous mapping $F : S^1 \to S^1$ has degree n if and only if F is homotopic to the map $z \mapsto z^n$ of S^1 onto itself.

Solution. (i) This is equivalent to saying that two continuous closed paths in $\gamma, \delta : [0,1] \to \mathbf{R}^2 \setminus \{0\}$ are homotopic is and only if $W(\gamma, 0) = W(\delta, 0)$. This was done in Homework 5.

(ii) You also showed in Homework 5 that the mapping $z \mapsto z^n$ of the unit circle has degree n. Thus (ii) follows at once from (i). □

Problem 3. (i) Let $F, F' : X \to Y$ and $G, G' : Y \to Z$ be continuous mappings. Prove that if F is homotopic to F', and G is homotopic to G', then the composite $G \circ F$ is homotopic to $G' \circ F'$.

(ii) Let F, G be continuous mappings from the unit circle S^1 into itself. Prove that

$$\deg(G \circ F) = \big(\deg F\big) \cdot \big(\deg G\big).$$

Solution. (i) If $H : X \times [0,1] \to Y$ is a homotopy from F to F' and $K : Y \times [0,1] \to Z$ is a homotopy from G to G' (as in the definition above), then define $\Gamma : X \times [0,1] \to Z$ by

$$\Gamma(x,s) = K(H(x,s),s).$$

Then Γ is continuous because H and K are both continuous,

$$\Gamma(x,0) = K(H(x,0),0) = K(F(x),0) = G(F(x)) = G \circ F(x),$$

and

$$\Gamma(x,0) = K(H(x,1),1) = K(F'(x),1) = G'(F'(x)) = G' \circ F'(x).$$

(ii) If $\deg F = n$, then F is homotopic to $z \mapsto z^n$, and if $\deg G = m$, then G is homotopic to $z \mapsto z^m$. By (i), the composite $G \circ F$ is homotopic to the composition of the nth power map and the mth power map, that is to $z \mapsto z^{nm}$, which has degree nm. □

Problem 4. Suppose that F is a continuous mapping from the positive octant $\{(x,y,z) \mid x \geq 0, y \geq 0, z \geq 0\}$ to itself. Show that there is a unit vector P in this octant, and a nonnegative number λ, such that $F(P) = \lambda P$.

Solution. The positive octant intersects the unit sphere in a triangular sector T. Define a mapping G form T into T by $G(P) = F(P)/|F(P)|$. Because T is homeomorphic to a disk (it is a challenging exercise for you to describe one such homeomorphism), there is a point P in T such that $G(P) = P$. Then $F(P) = \lambda P$ with $\lambda = |F(P)|$, as desired. □

Problem 5. Let $f : C \to C'$ be a continuous mapping between circles.

(i) Prove that if $f(P^*) = f(P)$ for all P, then the degree of f is even.

(ii) Prove that if $f(P^*) \neq f(P)$ for all P, then f is surjective.

Solution. There is no loss of generality if we assume that $C = C' = S^1$ is the unit circle. (Recall that the antipode of a point P in the unit circle is $P^* = -P$.
(i) The degree of f is the winding number $W(\gamma,0)$ of the path $\gamma(t) = F(\cos t, \sin t)$ $(0 \leq t \leq 2\pi)$ around the origin. Let δ_1 be the path $\delta_1(t) = F(\cos t, \sin t)$ $(0 \leq t \leq \pi)$ and δ_2 be the path $\delta_2(t) = F(\cos(t+\pi), \sin(t+\pi))$ $(0 \leq t \leq \pi)$. Then $\delta_2(t) = F(-\cos t, -\sin t) = F(\cos t, \sin t) = \delta_1(t)$, so $W(\delta_2,0) = W(\delta_1,0)$. But $W(\gamma,0) = W(\delta_1,0) + W(\delta_2,0) = 2W(\delta_1,0)$, an even number.
(ii) If $f(P) \neq f(P^*)$ for all P in C, define

$$g(P) = \frac{f(P) - f(P*)}{|f(P) - f(P*)|}.$$

Then $g(P^*) = -g(P)$, so that $\deg(g)$ is odd; in particular $\deg(g) \neq 0$.

Now g is homotopic to f. Indeed, because for every point P in the unit circle the point $f(P)$ is in the unit circle and $f(P) \neq f(P^*)$, the vector $f(P) - sf(P*) \neq 0$. Therefore

$$H(P,s) = \frac{f(P) - sf(P^*)}{|f(P) - sf(P*)|}$$

defines a continuous mapping $H : S^1 \times [0,1] \to S^1$ that satisfies $H(P,1) = g(P)$ and $H(P,0) = f(P)/|f(P)| = f(P)$ for all P in S^1.

By Problem 2, the mappings f and g have the same degree, and so $\deg(f) \neq 0$ also. By Problem 1, f must be surjective. □