Math 550. Homework 5. Solutions

Problem 1. Let $\gamma\colon [a,b] \to \mathbf{R}^2 \setminus \{0\}$ be a continuous path. Prove that there are continuous functions $r : [a,b] \to \mathbf{R}_+$ (the positive real numbers) and $\theta : [a,b] \to \mathbf{R}$, so that

$$\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t)), \qquad a \le t \le b. \tag{$*$}$$

Prove also that r is uniquely determined, and θ is uniquely determined up to an integer multiple of 2π.

Hint. Show in fact that $r(t) = |\gamma(t)|$, and that if γ^t denotes the restriction of γ to the interval $[a,t]$, for $a \le t \le b$, and θ_a is an angle for $\gamma(a)$, then one may take

$$\theta(t) = \theta_a + 2\pi W(\gamma^t, 0).$$

Solution. The function $r(t) = |\gamma(t)|$ is continuous because γ is continuous and the norm of a vector is continuous.

We show that $\theta(t) = \theta_a + 2\pi W(\gamma^t, 0)$ is continuous and satisfies $(*)$. Here θ_a is an angle for $P_a = \gamma(a)$.

Suppose first that $\gamma[a,b]$ is contained in a single sector U with angle function θ_U. The, for any t in $[a,b]$, the image of the path γ^t is also contained in U because $\gamma^t[a,t] = \gamma([a,t])$. Therefore U and θ_U can be used as sector and angle function for computing the winding number $W(\gamma^t, 0)$ of γ^t around 0,

$$\begin{aligned} W(\gamma^t, 0) &= \frac{1}{2\pi}\left\{\theta_U(\gamma^t(t)) - \theta_U(\gamma^t(a))\right\} \\ &= \frac{1}{2\pi}\left\{\theta_U(\gamma(t)) - \theta_U(\gamma(a))\right\}. \end{aligned}$$

It follows from this equation that $W(\gamma^t, 0)$ is a continuous function of t in $[a,b]$, because γ and θ_U are continuous.

The general case is similar and is left to you.

Now we have to show: (1) these functions $r(t)$ and $\theta(t)$ satisfy $(*)$, (2) that $r(t)$ is unique, and (3) that $\theta(t)$ is unique up to an integer multiple of 2π. The only part that requires some comment is (3). Suppose that there is another continuous function $\mu(t)$ such that $\gamma(t) = (r(t)\cos\mu(t), r(t)\sin\mu(t))$. Then for each t, both $\mu(t)$ and $\theta(t)$ are angles for $\gamma(t)$, and thus $\mu(t) - \theta(t) = 2\pi n(t)$ for some integer $n(t)$. (This integer $n(t)$ may depend on t.) But if μ and θ are continuous on $[a,b]$, then the function $t \mapsto n(t)$ is also continuous on $[a,b]$. But an integer valued, continuous function on an interval must be identically constant. That is, there is an integer n such that $n(t) = n$ for all t in $[a,b]$.

$\square$

Definition 1. Let $\gamma\colon [a,b] \to \mathbf{R}^2 \setminus \{0\}$ be a continuous path. By Problem 1, for any choice of angle θ_a for the initial point $P_a = \gamma_a$, there is a unique continuous path $\widetilde{\gamma}\colon[a,b] \to \{(r,\theta) \mid r > 0$ such that $p \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(a) = (r(a), \theta_a)$, where p is the polar coordinate mapping at the origin, $p(r,\theta) = (r\cos\theta, r\sin\theta)$. Such path $\widetilde{\gamma}$ is called a lifting of γ with starting point $(r(a), \theta_a)$.

Problem 2. Let $\gamma, \delta : [a,b] \to \mathbf{R}^2 \setminus \{0\}$ be two continuous paths with the same endpoints. Prove that the following are equivalent:

(i) γ and δ are homotopic in $\mathbf{R}^2 \setminus \{0\}$ relative to endpoints;

(ii) $W(\gamma, 0) = W(\delta, 0)$; and

(iii) if $\widetilde{\gamma}$ and $\widetilde{\delta}$ are liftings of γ and δ with the same initial point (as in Definition 1 above), then $\widetilde{\gamma}$ and $\widetilde{\delta}$ have the same final point.

Solution. (i)⇒(ii) This was done in class. (ii)⇒(iii) If $\widetilde{\gamma}$ is a lift of γ to $\{(r,\theta) \mid r>0\}$ starting at (r_a,θ_a) and ending at (r_b,θ_b), then the winding number $W(\gamma,0)=\theta_b-\theta_a$ (refer to Problem 1). Thus if $\widetilde{\gamma}$ and $\widetilde{\delta}$ are lifts of γ and δ that start at the same point (r_a,θ_a), and if $W(\gamma,0)=W(\delta,0)$, then they must end at points of the form $(r_{b,\gamma},\theta_b)$ and $(r_{b,\delta},\theta_b)$, respectively. But γ and δ have the same final point, so $\gamma(b)=\delta(b)$, and this implies that $r_{b,\gamma}=|\gamma(b)|$ and $r_{b,\delta}=|\delta(b)|$.

(iii)⇒(i) If $\widetilde{\gamma}$ and $\widetilde{\delta}$ are paths in $R=\{(r,\theta) \mid r>0\}$ with the same endpoints, they are homotopic in R because R is convex. If $\widetilde{H}:[a,b]\times[0,1]\to R$ is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\delta}$, then the composite $H: p\circ\widetilde{H}$ (where $p(r,\theta)=(r\cos\theta, r\sin\theta)$ is the polar coordinate map) is a homotopy from γ to δ in $\mathbf{R}^2\setminus\{0\}$ (Verify!). ☐

Problem 3. Identify $\mathbf{R}^2$ with the complex numbers $\mathbf{C}$, so that the vectors (x,y) corresponds to the complex number $z=x+iy$. Let C be the unit circle $\{|z|=1\}$ in $\mathbf{C}$. Determine the winding number $W(F,0)$ for the following mappings $F:C\to\mathbf{C}$.

(i) $F(z)=z^n$, n an integer.

(ii) $F(z)=-z$.

(iii) $F(z)=\bar{z}$.

Solution. From class, $\deg F=W(\gamma,0)$, where $\gamma(t)=F(\cos t,\sin t)$.

(i) In this case, $\gamma(t)=(\cos t+i\sin t)^n=(\cos nt,\sin nt)$. This is a smooth path so the winding number

$$W(\gamma,0)=\frac{1}{2\pi}\int_\gamma \omega_\theta = n.$$

(ii) Here $\gamma(t)=(-\cos t,-\sin t)$ and

$$W(\gamma,0)=\frac{1}{2\pi}\int_\gamma \omega_\theta = -1$$

(iii) Here $\gamma(t)=(\cos t,-\sin t)$ and $W(\gamma,0)=1$. ☐

Problem 4. Let C be the circle centered at the origin, and let $F:C\to\mathbf{R}^2$ be a continuous mapping such that the vector $F(P)$ is never tangent to the curve C at P, i.e., the dot product $P\cdot F(P)\neq 0$ for all P in C. Show that $W(F,0)=1$.

Solution. Suppose that C is the unit circle, so that $W(F,0)$ is the winding number $W(\gamma,0)$ of the path $\gamma(t)=F(\cos t,\sin t)$.

The condition $F(P)\cdot P\neq 0$ for all P means that $F(P)$ is never on the line through the origin perpendicular to P. Since the circle C is connected, this implies that either the segment from P to $F(P)$ does not contain the origin for all P in C, or that the segment from $-P$ to $F(P)$ does not contain the origin for all P in C. Note that it also implies that F maps the circle into $R^2\setminus\{0\}$.

If it is the case that the segment from P to $F(P)$ does not contain the origin for all P in C, then define $H(t,s)=sF(\cos t,\sin t)+(1-s)(\cos t,\sin t)$, for $0\le t\le 2\pi$, $0\le s\le 1$. This function H is continuous and takes values on $\mathbf{R}^2\setminus\{0\}$, because if $H(t,s)=0$ for some (t,s) in $[0,2\pi]\times[0,1]$, then the segment from $P=(\cos t,\sin t)$ to $F(P)=F(\cos t,\sin t)$ would contain the origin.

Thus we obtained a map $H:[0,2\pi]\times[0,1]\to\mathbf{R}^2\setminus\{0\}$. This map is a homotopy from $\gamma(t)=F(\cos t,\sin t)$ to the path $\delta(t)=(\cos t,\sin t)$. Since this path δ has $W(\delta,0)=1$, we obtain

$$\deg F=W(\gamma,0)=W(\delta,0)=1.$$

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