Math 550. Homework 5. Due 10/15/03

Problem 1. Let $\gamma : [a,b] \to \mathbf{R}^2 \setminus \{0\}$ be a continuous path. Prove that there are continuous functions $r : [a,b] \to \mathbf{R}_+$ (the positive real numbers) and $\theta : [a,b] \to \mathbf{R}$, so that

$$\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t)), \qquad a \le t \le b.$$

Prove also that r is uniquely determined, and θ is uniquely determined up to an integer multiple of 2π.

Hint. Show in fact that $r(t) = |\gamma(t)|$, and that if γ^t denotes the restriction of γ to the interval $[a,t]$, for $a \le t \le b$, and θ_a is an angle for $\gamma(a)$, then one may take

$$\theta(t) = \theta_a + 2\pi W(\gamma^t, 0).$$

Definition 1. Let $\gamma : [a,b] \to \mathbf{R}^2 \setminus \{0\}$ be a continuous path. By Problem 1, for any choice of angle θ_a for the initial point $P_a = \gamma_a$, there is a unique continuous path $\tilde{}:[a,b] \to \{(r,\theta) \mid r > 0$ such that $p \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(a) = (r(a), \theta_a)$, where p is the polar coordinate mapping at the origin, $p(r,\theta) = (r\cos\theta, r\sin\theta)$. Such path $\widetilde{\gamma}$ is called a lifting of γ with starting point $(r(a), \theta_a)$.

Problem 2. Let $\gamma, \delta : [a,b] \to \mathbf{R}^2 \setminus \{0\}$ be two continuous paths with the same endpoints. Prove that the following are equivalent:

(i) γ and δ are homotopic in $\mathbf{R}^2 \setminus \{0\}$ relative to endpoints;

(ii) $W(\gamma, 0) = W(\delta, 0)$; and

(iii) if $\widetilde{\gamma}$ and $\widetilde{\delta}$ are liftings of γ and δ with the same initial point (as in Definition 1 above), then $\widetilde{\gamma}$ and $\widetilde{\delta}$ have the same final point.

Problem 3. Identify $\mathbf{R}^2$ with the complex numbers $\mathbf{C}$, so that the vectors (x,y) corresponds to the complex number $z = x + iy$. Let C be the unit circle $\{|z| = 1\}$ in $\mathbf{C}$. Determine the winding number $W(F,0)$ for the following mappings $F : C \to \mathbf{C}$.

(i) $F(z) = z^n$, n an integer.

(ii) $F(z) = -z$.

(iii) $F(z) = \bar{z}$.

Problem 4. Let C be the circle centered at the origin, and let $F : C \to \mathbf{R}^2$ be a continuous mapping such that the vector $F(P)$ is never tangent to the curve C at P, i.e., the dot product $P \cdot F(P) \ne 0$ for all P in C. Show that $W(F,0) = 1$.