Math 550. Homework 4. Solutions

Problem 1. Given a 1-form ω on an open set U, prove that the following are equivalent (i) $d\omega = 0$; (ii) $\displaystyle\int_{\partial R} \omega = 0$ for all closed rectangles R contained in U; (iii) every point in U has a neighborhood such that $\displaystyle\int_{\partial R} \omega = 0$ for all closed rectangles contained in the neighborhood. Is the same true if closed rectangles are replaced by disks?

Solution. (i) $\Rightarrow$ (ii) This is Green's theorem for a rectangle. (ii) $\Rightarrow$ (iii) By definition of neighborhood, the open set U is a neighborhood of each of its points. (iii) $\Rightarrow$ (ii) We show that every point in U has a neighborhood on which $\omega = df$. This implies that ω is closed, because d is a local operator: to know the value of $d\omega$ at a point x we only need to know ω in a neighborhood of P (d computes partial derivatives). By (iii), each $P = (x, y)$ in U has a neighborhood D such that $\int_R \omega = 0$ for every closed rectangle in D. This D can be take to be an open rectangle of the form $(x - \varepsilon, x + \varepsilon) \times (y - \varepsilon, y + \varepsilon)$. Thus we are (almost) in the situation of Prop 1.12, except that we don't know that ω is closed. But if you look carefully at the proof of Prop 1.12, you will notice that all that it requires is that the integral $\int_R \omega = \int_{\partial R} \omega = 0$, and this is what (iii) says. $\square$

Problem 2. Let $H : U \to \mathbf{R}^2$ be a smooth function. Show that $dH^* = H^* d$.

Solution. We need to do some calculations with partial derivatives, so write $H(u, v) = (x(u, v), y(u, v))$ and the coordinates x, y are functions of the coordinates u, v.

First we do the most elementary case, that is, we show that $dH^*x = H^*dx$ and $dH^*y = H^*dy$ for the coordinate functions. By definition, $H^*dx = (\partial x/\partial u)du + (\partial x/\partial v)dv$. But if we think of x as a function of u, v, then we see that $d(x \circ H) = (\partial x/\partial u)du + (\partial y/\partial v)dv$. Thus $H^*dx = d(x \circ H) = dH^*x$. Similarly, $H^*dy = dH^*y$.

For the general case, suppose that $f = f(x, y)$ is a function in the x, y coordinates. Then the chain rule applied to the composite $f \circ H(u, v)$ says that

$$\frac{\partial f \circ H}{\partial u} = \left(\frac{\partial f}{\partial x} \circ H\right)\frac{\partial x}{\partial u} + \left(\frac{\partial f}{\partial y} \circ H\right)\frac{\partial y}{\partial u}$$

and

$$\frac{\partial f \circ H}{\partial v} = \left(\frac{\partial f}{\partial x} \circ H\right)\frac{\partial x}{\partial v} + \left(\frac{\partial f}{\partial y} \circ H\right)\frac{\partial y}{\partial v}$$

Now, on the one hand, by the definition of H^* on functions and of the operator d given in class, and by the chain rule above, we have

$$\begin{aligned} dH^* f = d(f \circ H) &= \frac{\partial (f \circ H)}{\partial u} du + \frac{\partial (f \circ H)}{\partial v} dv \\ &= \left[\left(\frac{\partial f}{\partial x} \circ H\right)\frac{\partial x}{\partial u} + \left(\frac{\partial f}{\partial y} \circ H\right)\frac{\partial y}{\partial u}\right] du + \left[\left(\frac{\partial f}{\partial x} \circ H\right)\frac{\partial x}{\partial v} + \left(\frac{\partial f}{\partial y} \circ H\right)\frac{\partial y}{\partial v}\right] dv \end{aligned}$$

On the other hand, df is a 1-form, and by the definition of H^* on 1-forms we have

$$\begin{aligned}
H^*df &= H^*(\partial f/\partial x dx + \partial f/\partial y dy) \\
&= \left(\frac{\partial f}{\partial x}\circ H\right)H^*(dx) + \left(\frac{\partial f}{\partial y}\circ H\right)H^*(dy) \\
&= \left(\frac{\partial f}{\partial x}\circ H\right)\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \left(\frac{\partial f}{\partial y}\circ H\right)\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) \\
&= \left[\left(\frac{\partial f}{\partial x}\circ H\right)\frac{\partial x}{\partial u} + \left(\frac{\partial f}{\partial y}\circ H\right)\frac{\partial y}{\partial u}\right]du + \left[\left(\frac{\partial f}{\partial x}\circ H\right)\frac{\partial x}{\partial v} + \left(\frac{\partial f}{\partial y}\circ H\right)\frac{\partial y}{\partial v}\right]dv
\end{aligned}$$

I let you do, or revise, the proof of $dH^*\omega = H^*d\omega$ for a 1-form ω. It is pretty similar to the one for 0-forms. □

Problem 3. Let R be a region in the plane between two concentric circles γ_1 and γ_2 of radius $r_1 < r_2$. Prove that if U is an open set containing R and ω is a 1-form on U, then

$$\int_R d\omega = \int_{\partial R} \omega,$$

where $\partial R = \gamma_2 - \gamma_1$.

Solution. We may assume without loss of generality that the center of these circles is the origin. If the 1-form ω is given by $\omega = p(x,y)dx + q(x,y)dy$, and if you parametrize $\gamma_1(t) = (r_1\cos t, r_1\sin t)$ and $\gamma_2(t) = (r_2\cos t, r_2\sin t)$, $0 \le t \le 2\pi$, then

$$\int_{\gamma_1}\omega = \int_0^{2\pi}\Big(-p(r_1\cos t, r_1\sin t)r_1\sin t + q(r_1\cos t, r_1\sin t)r_1\cos t\Big)dt$$

and

$$\int_{\gamma_2}\omega = \int_0^{2\pi}\Big(-p(r_2\cos t, r_2\sin t)r_2\sin t + q(r_2\cos t, r_2\sin t)r_2\cos t\Big)dt$$

If $H : [0,2\pi]\times[r_1,r_2] \to R$ is the mapping given by

$$H(t,s) = (s\cos t, s\sin t),$$

then the coordinates x,y are given as functions of t,s by

$$x(t,s) = s\cos t, \quad \text{and} \quad y(t,s) = s\sin t$$

and

$$\frac{\partial x}{\partial t} = -s\sin t, \quad \frac{\partial x}{\partial s} = \cos t, \quad \frac{\partial y}{\partial t} = s\cos t, \quad \text{and} \quad \frac{\partial y}{\partial s}\sin t,$$

Therefore the 1-form $H^*\omega$ is

$$\begin{aligned}
H^*\omega &= \left(p(x(t,s),y(t,s))\frac{\partial x}{\partial t} + q(x(t,s),y(t,s))\frac{\partial y}{\partial t}\right)dt + \left(p(x(t,s),y(t,s))\frac{\partial x}{\partial s} + q(x(t,s),y(t,s))\frac{\partial y}{\partial s}\right)ds \\
&= \Big(-p(s\cos t, s\sin t)s\sin t + q(s\cos t, s\sin t)s\cos t\Big)dt + \Big(p(s\cos t, s\sin t)\cos t + q(s\cos t, s\sin t)\sin t\Big)ds.
\end{aligned}$$

By applying Green's theorem for a rectangle to $H^*\omega$ on $K = [0,2\pi]\times[r_1,r_2]$, using Problem 2, and because ω is closed, we have

$$\int_{\partial K} H^*\omega = \int_K dH^*\omega = \int_K H^*d\omega = 0.$$

Now we compute $\int_{\partial K} H^*\omega$. This is the sum of four terms, one for each side of K, as follows. Parametrize the bottom side as $\delta_1(t) = (t, r_1)$, $0 \le t \le 2\pi$; the right side as $\delta_2(s) = (2\pi, s)$, $r_1 \le s \le r_2$; the top side as $\delta_3(t) = (t, r_2)$, $0 \le t \le 2\pi$; and the left side as $\delta_2(s) = (0, s)$, $r_1 \le s \le r_2$. Then

$$\int_{\partial K} H^*\omega = \int_{\delta_1} H^*\omega + \int_{\delta_2} H^*\omega - \int_{\delta_3} H^*\omega - \int_{\delta_4} H^*\omega.$$

We compute each of these four terms separately, taking into account the expression for $H^*\omega$ above.

$$\int_{\delta_1} H^*\omega = \int_0^{2\pi} \Big(-r_1 p(r_1 \cos t, r_1 \sin t) \sin t + r_1 q(r_1 \cos t, r_1 \sin t) \cos t \Big) dt = \int_{\gamma_1} \omega,$$

$$\int_{\delta_2} H^*\omega = \int_{r_1}^{r_2} q(s, 0)\, ds = \int_{\delta_3} H^*\omega,$$

and

$$\int_{\delta_2} H^*\omega = \int_0^{2\pi} \Big(-r_2 p(r_2 \cos t, r_2 \sin t) \sin t + r_2 q(r_2 \cos t, r_2 \sin t) \cos t \Big) dt = \int_{\gamma_2} \omega.$$

Therefore,

$$0 = \int_{\partial K} H^*\omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega.$$

□

Problem 4. Prove that the relation of being homotopic relative to endpoints, or homotopic as closed paths, is an equivalence relation.

Solution. Suppose that $\gamma, \delta, \sigma : [a,b] \to U$ are smooth paths in U. Denote the relation of being homotopic (relative to endpoints or as closed paths) by $\sim$.
(i) $\gamma \sim \gamma$. Take $H(t,s) = \gamma(t)$ for all s in $[0,1]$ and all t in $[a,b]$.
(ii) If $\gamma \sim \delta$, then $\delta \sim \gamma$. Suppose that $H : [a,b] \times [0,1] \to U$ is a homotopy from γ to δ. Then $K(t,s) = H(t, 1-s)$ defines a homotopy from δ to γ.
(iii) If $\gamma \sim \delta$ and $\delta \sim \sigma$, then $\gamma \sim \sigma$. This was actually too difficult because we are working in the smooth category (my bad). Suppose that H is a homotopy from γ to δ and K is one from δ to σ. The idea is to combine these two homotopies, like this

$$N(t,s) = \begin{cases} H(t, 2s), & 0 \le s \le 1/2 \\ K(t, 2s-1), & 1/2 \le s \le 1 \end{cases}$$

This N has all the properties required for being a homotopy from γ to σ, except that it may not be smooth (it may fail to have partial derivatives along the horizontal segment $s = 1/2$, because there both H and K intervene). This idea can nevertheless be made to work, but the details are rather technical and so it will be discussed in class at a latter time. It is nevertheless a complete proof in the setting of continuous homotopies between continuous paths. □

Problem 5. Let $\gamma : [a,b] \to \mathbf{R}^2 \setminus \{P\}$ be a continuous path, and let ν be a vector in the plane. Let $\gamma + \nu$ denote the path in $\mathbf{R}^2 \setminus \{P + \nu\}$ defined by $(\gamma + \nu)(t) = \gamma(t) + \nu$. Prove that

$$W(\gamma + \nu, P + \nu) = W(\gamma, P).$$

Solution. To compute $W(\gamma, P)$ we first choose a partition $a = t_0 < t_1 < \cdots < t_n = b$ so that each image $\gamma[t_{j-1}, t_j]$ is contained in a sector U_j at P, and then we chose angle functions θ_j for these sector.

Given this, it is clear $U_j + \nu$ are sectors at $P + \nu$ and that the partition $a = t_0 < t_1 < \cdots < t_n = b$ also has the property that the image $(\gamma + \nu)[t_{j-1}, t_j] \subset U_j + \nu$. Moreover, the function $\theta'_j(Q) = \theta_j(Q - \nu)$ is an angle function for the sector $U_j + nu$. We then have (with $P_j = \gamma(t_j)$

$$
\begin{aligned}
W(\gamma+\nu, P+\nu) &= \frac{1}{2\pi}\sum_{j=1}^{n}\big(\theta'_j(P_j+\nu) - \theta'_j(P_{j-1}+\nu)\big) \\
&= \frac{1}{2\pi}\sum_{j=1}^{n}\big(\theta_j(P_j) - \theta_j(P_{j-1})\big) \\
&= W(\gamma, P)
\end{aligned}
$$

□