Math 550. Homework 2. Solutions

Problem 1. Let U be the union of two open sets U_1, U_2, i.e., $U = U_1 \cup U_2$. Let f_1 and f_2 be smooth functions on U_1 and U_2, respectively, such that $f_1(x) = f_2(x)$ for every x in $U_1 \cap U_2$. Prove that

$$f(x) = \begin{cases} f_1(x) & x \in U_1, \\ f_2(x) & x \in U_2 \end{cases}$$

is a smooth function on U.

Solution. Since $f_1 = f_2$ on $U_1 \cap U_2$ and $U = U_1 \cup U_2$, f is a well-de£ned function on U, that is, it assigns one and only one value $f(x)$ to each x in U.

The function f is also smooth. This is so because 'to be smooth' is a local property, and f agrees locally with either f_1 or f_2, which are smooth functions. (To agree locally means that each point in U has a neighborhood where f agrees with f_1 or with f_2. Can you write a formal proof of this fact?) □

Problem 2. Let $\gamma : [a,b] \to U$ be a path in U and de£ne $\gamma^{-1} : [a,b] \to U$ by $\gamma^{-1}(t) = \gamma(b+a-t)$. Prove that

$$\int_{\gamma^{-1}} \omega = -\int_{\gamma} \omega$$

for every 1-form ω on U

Solution. If $\gamma(t) = (x(t), y(t))$, then $\gamma^{-1}(t) = (\tilde{x}(t), \tilde{y}(t))$, where

$$\tilde{x}(t) = x(a+b-t) \qquad \text{and} \qquad \tilde{y}(t) = y(b+a-t).$$

The de£nition of integral of a 1-form along a path says that

$$\int_{\gamma^{-1}} \omega = \int_a^b \big(p(\tilde{x}(t), \tilde{y}(t))\tilde{x}'(t) + q(\tilde{x}(t), \tilde{y}(t))\big)\tilde{y}'(t)dt.$$

Now $\tilde{x}'(t) = -x(b+a-t)$ and $\tilde{y}' = -y(b+a-t)$, and substituting above we obtain

$$\int_{\gamma^{-1}} \omega = -\int_a^b \big(p(x(b+a-t), y(b+a-t))x'(b+a-t) + q(x(b+a-t), y(b+a-t))y'(b+a-t)\big)dt.$$

Finally do the change of variable $s = b+a-t$ and get

$$\int_{\gamma^{-1}} \omega = -\int_b^a p((x(s), y(s))x'(s) + q(x(s), y(s))(-ds).$$

After canceling out the two negative signs and switching the limits of integration, this becomes $-\int_\gamma \omega$. □

Problem 3. Let U be an open disk in the plane, i.e., $U = \{(x,y) \mid (x-x_0)^2 + (y-y_0)^2 < r^2\}$. Prove that if ω is a closed 1-form on U, then there is a smooth function f on U such that $df = \omega$.

Show that this is also true if U is a star-shaped open region. This means that there is a point P_0 in U such that for any other point P in U the segment PP_0 is contained in U.

Solution. Suppose that U is a disk with center $P_0 = (x_0, y_0)$. If P is any other point in this disk, then the rectangle with corners P_0 and P is contained in U, and you can apply verbatim the proof of Prop 1.12 in the book. You should make sure that you understand this proof.

□

Problem 4. Let U be an union of open sets $U_1, U_2, \cdots, U_n$, and let ω be a 1-form on U such that the restriction of ω to each U_j is exact. Prove that if $(U_1 \cup \cdots \cup U_{j-1}) \cap U_j$ is connected for $1 < j \leq n$, then ω is exact on U.

Note. There was a typo in my handout. You all get full credit for this problem.

Solution. This is most easily proved by induction. In class we saw that if V and W are two open sets with $V \cap W$ connected, and ω is a 1-form on $U \cup V$ that is exact on each of U and V, then ω is exact on $U \cup V$.

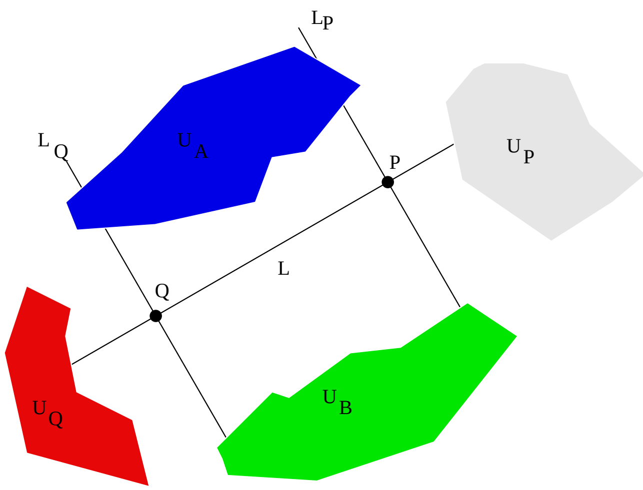
Suppose that the statement has been proved for the union of $n-1$ open sets. Let U be as in the statement. Then by the induction hypothesis, the 1-form ω is exact on $U_1 \cup \cdots \cup U_{n-1}$ and on U_n. Since the intersection $(U_1 \cup \cdots \cup U_{n-1}) \cap U_n$ is connected, it is exact on $U_1 \cup \cdots \cup U_n$. □

Problem 5. For any point $P = (x_0, y_0)$, let $\omega_{P,\theta}$ be the 1-form on $\mathbf{R}^2 \setminus \{P\}$ de£ned by

$$\omega_{P,\theta} = \frac{1}{(x-x_0)^2 + (y-y_0)^2} \Big(-(y-y_0)dx + (x-x_0)dy \Big).$$

Prove that for any two points P and Q, the 1-form $\omega = \omega_{P,\theta} - \omega_{Q,\theta}$ is exact on $\mathbf{R}^2 \setminus L$, where L is the line segment from P to Q.

Solution. Let ℓ be the line through P and Q, and let ℓ_P be the lines perpendicular to ℓ through P and Q, respectively. Let U_P be the (open) half plane determined by ℓ_P not containing Q, let U_Q be the half plane determined by ℓ_Q not containing P, and let U_A and U_B be the two half planes determined by ℓ. It is clear that $U_P \cup U_A \cup U_Q \cup U_B = \mathbf{R}^2 \setminus L$. These con£guration of half planes looks like the following Figure.



Because the form ω is closed and each of U_P, U_A, U_Q, U_B is an open rectangle, the equation $df = \omega$ can be solved on each of them.

On U_P there is a function f_P such that $df_P = \omega$. On U_A there is a function f_A such that $df_A = \omega$. Since the intersection $U_A \cap U_P$ is connected, the functions f_A and f_P differ by constant there, and we may assume this constant is 0, that is, that $f_A = f_P$ on $U_A \cap U_P$.

On U_Q there is a function f_Q such that $df_Q = \omega$ there, and since $U_Q \cap U_A$ is connected, we may, as before, take $f_Q = f_A$ on $U_A \cap U_Q$.

Finally, there is a function f_B on U_B such that $df_B = \omega$ there, and as before, we can make $f_B = f_Q$ on $U_B \cap U_Q$.

We are left to compare f_P and f_B on the intersection $U_B \cap U_P$. Since this open set is connected, and the difference $f_P - f_B$ satis£es $d(f_P - f_B) = \omega - \omega = 0$, we have that $f_P - f_B = c$ is constant there, and if we show that this constant is 0, we would have obtained a function f de£ned on $\mathbf{R}^2 \setminus L$ such that $df = \omega$.

A proposition in class (Prop 1.16) tells you that if γ is a circle that encloses both P and Q, then the integral

$$\int_\gamma \omega = c.$$

This proposition also implies the following. If γ is a circle (traveled anticlockwise) not containing P, then

$$\int_\gamma \omega_{P,\theta} = \begin{cases} 2\pi & \text{if } \gamma \text{ encloses } P, \\ 0 & \text{if } \gamma \text{ does not enclose } P. \end{cases}$$

Similarly, if γ is a circle (traveled clockwise) not containing Q, then

$$\int_\gamma \omega_{P,\theta} = \begin{cases} 2\pi & \text{if } \gamma \text{ encloses } P, \\ 0 & \text{if } \gamma \text{ does not enclose } P. \end{cases}$$

Therefore, if γ is a circle enclosing both P and Q, we have

$$\int_\gamma \omega = \int_\gamma \omega_{P,\theta} - \int_\gamma \omega Q, \theta = 2\pi - 2\pi = 0.$$

$\square$