Math 550. Homework 2 (Revised). Due 9/22/2003

Problem 1 Let U be the union of two open sets U_1, U_2, i.e., $U = U_1 \cup U_2$. Let f_j be a smooth functions on U_j, $j = 1, 2$, such that $f_1(x) = f_2(x)$ for every x in $U_1 \cap U_2$. Prove that

$$f(x) = \begin{cases} f_1(x) & x \in U_1, \\ f_2(x) & x \in U_2 \end{cases}$$

is a smooth function on U.

Problem 2 Let $\gamma : [a,b] \to U$ be a path in U and define $\gamma^{-1} : [a,b] \to U$ by $\gamma^{-1}(t) = \gamma(b+a-t)$. Prove that

$$\int_{\gamma^{-1}} \omega = -\int_{\gamma} \omega$$

for every 1-form ω on U

Problem 3 Let U be an open disk in the plane, i.e., $U = \{(x,y) \mid (x-x_0)^2 + (y-y_0)^2 < r^2\}$. Prove that if ω is a closed 1-form on U, then there is a smooth function f on U such that $df = \omega$.

Show that this is also true if U is a star-shaped open region. This means that there is apoint P_0 in U such that for any other point P in U the segment PP_0 is contained in U.

Problem 4 Let U be an union of open sets $U_1, U_2, \cdots, U_n$, and let ω be a 1-form on U such that the restriction of ω to each U_j is exact. Prove that if $(U_1 \cup \cdots \cup U_{j-1}) \cap U_j$ is connected for $1 < j \le n$, then ω is exact on U.

Problem 5 For any point $P = (x_0, y_0)$, let $\omega_{P,\theta}$ be the 1-form on $\mathbf{R}^2 \setminus \{P\}$ defined by

$$\omega_{P,\theta} = \frac{1}{(x-x_0)^2 + (y-y_0)^2} \Big(-(y-y_0)dx + (x-x_0)dy \Big).$$

Prove that for any two points P and Q, the 1-form $\omega = \omega_{P,\theta} - \omega_{Q,\theta}$ is exact on $\mathbf{R}^2 \setminus L$, where L is the line segment from P to Q.