Math 550. Homework 1. Solutions (Revised)

Problem 1. Let U be an open subset of the plane. Prove that U is connected if and only if every locally constant function on U is constant on U.

Solution. If U is connected and f is a locally constant function on U, then f must be continuous on U (verify!). If f is not constant, then it must take on at least two different real numbers, say $a \neq b$. Then $f^{-1}(\{a\})$ is an open non-empty subset of U ("open" because f is locally constant, "non-empty" because we just said that f attains the value a), and $f^{-1}(\mathbf{R} \setminus \{b\})$ is also a non-empty subset of U ("open" because $\mathbf{R} \setminus \{a\} = (-\infty, a) \cup (a, \infty)$ is open in $\mathbf{R}$ and f is continuous, "non-empty" because f attains the value b, which is in $\mathbf{R} \setminus \{a\}$). Furthermore, U is the disjoint union $U = f^{-1}(\{a\}) \cup f^{-1}(\mathbf{R} \setminus \{b\})$, contradicting that U is connected.

If U is not connected, then we can write $U = U_1 \cup U_2$, where U_1 and U_2 are open, non empty and disjoint open sets. Define f by $f(x) = 1$ if x is in U_1 and $f(x) = 2$ if x s in U_2. Then f is locally constant on U (verify!) but it is not constant.

2nd Solution. Another proof can be obtained by using a result proved in class: an open set U is connected if and only if every smooth function f on U such that $\partial f / \partial x = 0$ and $\partial f / \partial y = 0$ is constant on U.

Suppose that U is connected. If f is locally constant, then f is smooth and $\partial f / \partial x = \partial f / \partial y = 0$ because to compute the partial derivatives of f at a point P you only need to know the behavior of f in a small disk about P. Since f is locally constant, you can always take this disk to be such that f is actually constant on it.

For the converse, just note that any function f such that $\partial f / \partial x = \partial f / \partial y = 0$ is locally constant (this was shown in class). The hypothesis implies that f must be constant in U, so U is connected by the result in class quoted above.

$\square$

Problem 2. Let $\omega = y dx - x dy$ and let γ be the line segment from $(0,0)$ to $(1,1)$. Compute the path integral $\int_\gamma \omega$.

Solution. The line segment from $(0,0)$ to $(1,1)$ is given by $\gamma(t) = (t,t)$, $0 \leq t \leq 1$. Then $\int_\gamma \omega = \int_0^1 (t dt - t dt) = 0$. $\square$

Problem 3. Let f, g be two smooth functions on U. Prove that $df = dg$ on U if and only if $f - g$ is locally constant on U.

Solution. Taking partial derivatives is a linear operation, hence $d(f-g) = df - dg$. Thus $df = dg$ if and only if $df - dg = d(f-g) = 0$ if and only if $f - g$ is locally constant on U (by Proposition in class). $\square$

Problem 4. Let ω_θ be the 1-form on $\mathbf{R}^2 \setminus \{(0,0)\}$ given by $\omega_\theta = \frac{1}{x^2+y^2}\left(-y dx + x dy\right)$. On which of the following open sets U is there a smooth function g with $dg = \omega_\theta$ on U? (Prove your answers.) (i) The upper half plane $\{(x,y) \mid y > 0\}$. (ii) The union of the upper half plane and the right half plane. (iii) The annulus $\{(x,y) \mid 1 < x^2 + y^2 < 2\}$.

Solution. (i) Yes, take $g(x,y) = \cot^{-1}(x/y)$. This is smooth on $\{y > 0\}$ and satisfies $dg = \omega_\theta$ there.

(ii) Yes. There are two ways of writing a solution. (ii-1) In class we constructed a function f_1 on the right half plane U_1 such that $df_1 = \omega_\theta$ on U_1. Part (i) shows that there exists a function f_2 on the upper half plane such that $df_2 = \omega_\theta$ on U_2. Therefore, on $U_1 \cap U_2$ we have $df_1 = df_2$, that is, the function $f_1 - f_2 = c$ is constant on $U_1 \cap U_2$ (by Problem 3). Define a function g on $U_1 \cup U_2$ by setting $g(x) = f_1(x)$ if x is in U_1, and $g(x) = f_2(x) + c$ if x is in U. This makes sense because, if x is in $U_1 \cap U_2$, then $f_1(x) = f_2(x) + c$. Moreover, g is smooth and satisfies $dg = \omega_\theta$ on U. (If this sounds like trivial and a little tedious to write down, then you may be right. But you have to do this kind of thing it at least once!)

(ii-2) Let $P_0 = (1,1)$. For any point P in the union of the half planes, let γ_P be the line segment from P_0 to P and set $f(P) = \int_{\gamma_P} \omega_\theta$.

(iii) No. If γ is the path $\gamma(t) = ((3/2)\cos t, (3/2)\sin t)$, $0 \leq t \leq 2\pi$, then $\int_\gamma \omega_\theta = 2\pi$. But we know that if ω_θ was the differential of a function on the annulus, then $\int_\gamma \omega_\theta = 0$ because γ is a closed path. $\square$

Problem 5. Is $\omega = \frac{1}{(x^2+y^2)^2}\big(xdx+ydy\big)$ the differential of a function on $\mathbf{R}^2 \setminus \{(0,0)\}$?

Solution. Yes. Take $f(x,y) = \frac{1/2}{x^2+y^2}$ and check that $df = \omega$. $\square$