

Math 512A. 2nd Midterm Study Guide

The 2nd midterm will be similar to the first one in what concerns length and type of problems. That is, 4 or 5 problems, each with 2 parts: one of theoretical nature (like a definition or statement of a theorem) and one of a more applied nature.

The outline below, together with textbook, class notes, homeworks and answer sheets, should help you organize your study for this test. The questions included should help you test your understanding of the material.

A. Compact Sets

Compact sets. The Bolzano-Weierstrass Theorem: compact if and only if closed and bounded. Preservation (or not) of compactness under unions and intersections. Lower and upper bounds, supremum and infimum, and maximum and minimum of sets of real numbers.

- (A.1) The intersection of compact sets is compact.
- (A.2) Prove or give a counterexample: The union of compact sets is compact.
- (A.3) If $A \neq \emptyset$ is a subset of real numbers which is bounded above, then A has a supremum.
- (A.4) Find the supremum and infimum, if they exist, of the set $\{x \in \mathbf{R} \mid x < 0 \text{ and } x^2 + x - 1 < 0\}$. Does this set have a maximum? a minimum?
- (A.5) If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A . Then $B \neq \emptyset$, B is bounded above, and $\sup B = \inf A$.

B. Existence of Maximum

Continuous functions take compact sets to compact sets. A continuous function on an interval $[a, b]$ attains its maximum and minimum values.

- (B.1) Give an example of a function on $[0, 1]$ which is bounded but has neither maximum nor minimum.
- (B.2) Prove or give a counterexample: if f is continuous and K is compact, then $f^{-1}K$ is compact.
- (B.3) Suppose that $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous with $f(x) > 0$ for all x , and $\lim_{x \rightarrow \infty} f(x) = 0 = \lim_{x \rightarrow -\infty} f(x)$. Prove that f has a maximum.

Note. For any function f , we say that $\lim_{x \rightarrow \infty} f(x) = L$ if for every $\epsilon > 0$ there is M such that $|f(x) - L| < \epsilon$ when $x > M$, and we say that $\lim_{x \rightarrow -\infty} f(x) = L$ if for every $\epsilon > 0$ there is M such that $|f(x) - L| < \epsilon$ when $x < -M$.

- (B.4) Suppose that f is continuous on $[a, b]$ and let x be any number. Prove that there is a point on the graph of f which is closest to the point $(x, 0)$ on the plane; that is, prove that there is y in $[a, b]$ such that the distance from $(x, 0)$ to $(y, f(y))$ is \leq distance from $(x, 0)$ to $(z, f(z))$ for any z in $[a, b]$.
- (B.5) Prove that the assertion in (B.4) is false if $[a, b]$ is replaced by (a, b) , but is true if $[a, b]$ is replaced by \mathbf{R} .

C. Uniform Continuity

Uniform continuity: what it is and how it is different from continuity. Uniform continuity and compactness. Uniform continuity and Cauchy sequences.

- (C.1) Let $f(x) = x^a$. For which of the following values of a is f uniformly continuous on $[0, \infty)$: $a = 1/3, 1/2, 1, 2, 3$?
- (C.2) Find a function which is continuous and bounded on $[0, \infty)$, but not uniformly continuous on $[0, \infty)$.
- (C.3) Prove or give a counterexample: if f is continuous and bounded on $(0, 1]$, then f is uniformly continuous on $(0, 1]$.
- (C.4) Suppose that f is uniformly continuous on A , g is uniformly continuous on B , and $f(x)$ is in B for all x in A . Prove that $g \circ f$ is uniformly continuous on A .
- (C.5) Prove or give a counterexample: (i) If f is uniformly continuous, then f takes Cauchy sequences to Cauchy sequences; (ii) If f takes Cauchy sequences to Cauchy sequences, then f is uniformly continuous.

D. The Intermediate Value Theorem

Intermediate Value Theorem: statement, proof, applications. Continuous functions take intervals to intervals. Intervals are connected.

- (D.1) Prove or give a counterexample: (i) if $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and I is an interval, then $f(I)$ is also an interval; (ii) if $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and J is an interval, then $f^{-1}(J)$ is an interval.

- (D.2) A set A of real numbers is called *dense* if every nonempty open interval contains a point of A . (For example, the set of all rational numbers is dense in \mathbf{R} .) Prove that if $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and $f(x) = 0$ for all x in a dense set A , then $f(x) = 0$ for all x .
- (D.3) Prove that if f is continuous on an interval I and $f(x)$ is rational for all x in I , then f is constant on I .
- (D.4) How many continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$ are there which satisfy $(f(x))^2 = x^2$ for all x ?
- (D.5) Let f be continuous on $[a, b]$ and $f(a) < 0 < f(b)$. We proved in class that there is a smallest x in $[a, b]$ with $f(x) = 0$. Is there necessarily a second smallest x in $[a, b]$ with $f(x) = 0$? Prove that there is a largest x in $[a, b]$ with $f(x) = 0$.

E. The Cantor Set

What is the Cantor set? What are its topological properties? Closed? Bounded? Compact? Connected?

- (E.1) Prove that the Cantor set is compact.
- (E.2) Prove that the Cantor set is not connected.
- (E.3) Is there a continuous map of \mathbf{R} onto the Cantor set?
- (E.4) One of the constructions of the Cantor set involved removing the middle third interval of $], 1]$, then the two middle third intervals of the remaining intervals, and so on. What is the total length of the interval removed?

- (E.5) The Cantor set is uncountable. (This is a “diagonal argument” like that using in proving that $[0, 1]$ is uncountable, but now using the “base 3” decimal expansion instead of the “base 10” decimal expansion.)

F. The Derivative and The Mean Value Theorem

Definition of derivative. Calculations of derivatives, including the chain rule. Differentiability and continuity. The derivative and maxima and minima. Critical points of functions. Rolle’s theorem. Mean Value Theorem. Applications: estimates of functions, constant functions, increasing and decreasing.

- (F.1) If $f(x) = |x|^3$, find $f'(x)$ and $f''(x)$ (to be sure, f'' is the derivative of f'). Does $f'''(x)$ exist for all x ?
- (F.2) Prove that: (i) if f has a critical point at a and $f''(a) > 0$, then f has a local minimum at a ; (ii) if f has a local minimum at a and $f''(a)$ exists, then $f''(a) \geq 0$.
- (F.3) Prove that if f is differentiable on (a, b) and f' is bounded on (a, b) , then f is uniformly continuous on (a, b) .
- (F.4) Prove that if f is continuous at a and $|f|$ is differentiable at a , then f is also differentiable at a . How are $f'(a)$ and $|f'(a)|$ related?
- (F.5) Let $a \neq 0$ and let x be such that $(x + a)^n = x^n + a^n$. Prove that if n is even, then $x = 0$, and that if n is odd, then $x = 0$ or $x = -a$.