Math 512A. 2nd Midterm Solutions

Problem 1. (i) Define the concepts "upper bound" and "infimum" for a set of real numbers.

(ii) If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A. Prove that $B \neq \emptyset$, that B is bounded above, and that $\sup B = \inf A$.

Solution. (ii) The set $B \neq \emptyset$ because A is bounded below (any lower bound for A is in B). Because A is nonempty, there is a in A, and this a satisfies $y \leq a$ for all y in B. Because of this, $y \leq x$ for all x in A and all y in B, and thus $\sup B \leq \inf A$.

If $\sup B < \inf A$, then there is a number x such that $\sup B < x < \inf A$. The inequality $\sup B < x$ implies that $b < x$ for all b in B and thus that x is not in B. The inequality $x < \inf A$ implies that $x < a$ for all a in A, and thus that x is a lower bound for A. Therefore x is in B. But this contradicts the inequality $\sup B < x$ as noted. □

Problem 2. (i) Give an example of a continuous function on $(0, 1)$ which is bounded but attains neither a maximum value nor a minimum value.

(ii) Suppose that f is continuous on $\mathbf{R}$, and that for any number M there exists $\delta > 0$ such that $f(x) > M$ if $|x| > \delta$. Prove that f attains a minimum value.

Solution. (i) The function $f(x) = x$ for x in $(0, 1)$ has $\sup\{f(x) \mid 0 < x < 1\} = 1$ and $\inf\{f(x) \mid 0 < x < 1\} = 0$, but $0 < f(x) < 1$ for all x in $(0, 1)$.

(ii) Take $M = f(0)$ and let δ be such that $f(x) > f(0)$ for $|x| > \delta$. The function f is continuous on the compact interval $[-\delta, \delta]$, so f attains a minimum value in that interval; that is, there is y in $[-\delta, \delta]$ such that $f(x) \geq f(y)$ for all x in $[-\delta, \delta]$. In particular, $f(0) \geq f(y)$ and thus $f(x) \geq f(0) \geq f(y)$ for all x such that $|x| > \delta$ also.

Note. This proof was done in class for polynomials of even degree. □

Problem 3. (i) Define the concept "uniformly continuous function."

(ii) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $(1, \infty)$.

Proof. (ii) If $x, y > 1$, then $\dfrac{1}{\sqrt{x} + \sqrt{y}} < \dfrac{1}{1+1} = \dfrac{1}{2}$. Rationalizing

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &= \frac{|\sqrt{x} - \sqrt{y}|(\sqrt{x} + \sqrt{y})}{\sqrt{x} + \sqrt{y}} \\ &= \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \\ &< \frac{1}{2}|x - y|. \end{aligned}$$

Therefore, given $\epsilon > 0$, take $\delta = 2\epsilon$. If $|x - y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < \dfrac{1}{2} 2\epsilon = \epsilon$. □

Problem 4. (i) State the Bolzano-Weierstrass Theorem. (This includes defining the concepts in the statement of this theorem.)

(ii) Construct the Cantor set C, and prove that C is compact.

Problem 5. (i) State the Intermediate Value Theorem.

(ii) Prove that if f is continuous on an interval J and $f(x)$ is rational for any x in J, then f is constant on J.

Solution. (ii) If f is not constant on J, then there are numbers a and b in J such that $f(a) \neq f(b)$. The Intermediate Value Theorem implies that for any number y between $f(a)$ and $f(b)$ there is a number x between a and b such that $f(x) = y$. This contradicts the hypothesis that f takes on rational numbers only, because between $f(a)$ and $f(b)$ there is at least one number y which is not rational. □

Problem 6. (i) For a function f and a point a in the domain of f, define the concepts "f is differentiable at a" and "derivative of f at a."

(ii) Let $f(x) = |x|^3$. Find $f'(x)$ and $f''(x)$, and find all numbers x for which $f'''(x)$ exists. (To be sure, f'' is the derivative of f', and f''' is the derivative of f'', whenever they exist.)

Solution. (ii) Note that $f(x) = x^3$ if $x \geq 0$ and $f(x) = -x^3$ is $x < 0$. Thus we immediately have $f'(x) = 3x^2$ if $x > 0$ and $f'(x) = -3x^2$ if $x < 0$. For $x = 0$, we find

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|^3}{h} = \lim_{h \to 0} h|h| = 0.$$

and we obtain that $f'(x) = 3x|x|$.

From the above we find that $f''(x) = 6x$ if $x > 0$ and $f''(x) = -6x$ if $x < 0$. For $x = 0$, we compute $f''(0)$ using the definition of derivative:

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{3h|h|}{h} 0,$$

and we obtain $f''(x) = 6|x|$ for all x.

It follows by similar arguments that $f'''(x) = 6\dfrac{|x|}{x}$ if $x \neq 0$, and that $f'''(0)$ does not exist. □

Problem 7. (i) State Rolle's Theorem and the Mean Value Theorem.

(ii) Let $a \neq 0$ and n be even. Prove that the polynomial equation $x^n + a^n = (x + a)^n$ has exactly one (real) solution.

Solution. (ii) Note first that if n is even, then $n - 1 > 1$ is odd and the function $g(x) = nx^{n-1}$ is strictly increasing (in particular, one-one).

Let $f(x) = (x + a)^n - x^n - a^n$. If $f(x_0) = 0$ for some $x_0 \neq 0$, then $f'(c) = 0$ for some c between 0 and x_0, by Rolle's Theorem. But $f'(x) = n(x + a)^{n-1} - nx^{n-1}$, so that $f'(c) = 0$ implies that $g(c + a) = g(c)$, contradicting that g is one-one.

□