Math 512A. Homework 9. Solutions

Problem 1. (i) Suppose that $g(x) = f(x+c)$ for all x. Prove, starting from the definition of the derivative, that $g'(x) = f'(x+c)$ for all x.

(ii) Prove that if $g(x) = f(cx)$, then $g'(x) = c \cdot f'(cx)$.

(iii) Suppose that f is differentiable and periodic, with period a, i.e., $f(x+a) = f(x)$ for all x. Prove that f' is also periodic with period a.

(iv) (Not required) Prove that if f is even, i.e., $f(x) = f(-x)$, then $f'(x) = -f'(-x)$.

(v) (Not required) Prove that if f is odd, i.e., $f(-x) = -f(x)$, then $f'(x) = f'(-x)$.

Solution. (i) According to the definition,

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h+c)-f(x+c)}{h} \qquad \text{because } g(y) = f(y+h) \\ &= f'(x+c) \end{aligned}$$

(ii) By the definition

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(cx+ch)-f(cx)}{h} \\ &= \lim_{h\to 0} c\frac{f(cx+ch)-f(cx)}{ch} \\ &= c\lim_{k\to 0} \frac{f(cx+k)-f(cx)}{k} \qquad \text{make } k = ch \\ &= cf'(cx) \end{aligned}$$

(iii) Follows directly from (i). □

Problem 2. (i) Let $f(x) = x^2$ if x is rational, and $f(x) = 0$ if x is irrational. Prove that f is differentiable at 0.

(ii) Let f be a function such that $|f(x)| \le x^2$ for all x. Prove that f is differentiable at 0.

(iii) (Not required) Let $\alpha > 1$. Prove that if f satisfies $|f(x)| \le |x|^\alpha$, then f is differentiable at 0.

Solution. (i) We compute

$$\frac{f(0+h)-f(0)}{h} = \begin{cases} h, & h \text{ is rational} \\ 0, & h \text{ is irrational} \end{cases}$$

It follows that $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$, so that f is differentiable at 0 and $f'(0) = 0$.

(ii) It follows from $|f(x)| \le x^2$ that $-x^2 \le f(x) \le x^2$ and $f(0) = 0$, so that

$$-h \le \frac{f(h) - f(0)}{h} \le h$$

and hence that f is differeniable at 0 and $f'(0) = 0$. □

Problem 3. Suppose that a and b are two consecutive roots of the polynomial function f, but that a and b are not double roots, so that we can write $f(x) = (x - a)(x - b)g(x)$ where $g(a) \ne 0$ and $g(b) \ne 0$.

(i) Prove that $g(a)$ and $g(b)$ have the same sign.

(ii) Prove that there is some number x with $a < x < b$ and $f'(x) = 0$.

(iii) (Not required) Prove that (ii) holds true even if a and b are multiple roots. Hint: If $f(x) = (x-a)^n(x-b)^m g(x)$ where $g(a) \ne 0$ and $g(b) \ne 0$, consider the polynomial function $h(x) = f'(x)/(x - a)^{n-1}(x - b)^{m-1}$.

Solution. (i) If $g(a)$ and $g(b)$ have opposite sign, then, by the Intermediate Value Theorem, there is c in (a, b) such that $g(c) = 0$. Then $f(c) = 0$ also, contradicting that a and b are consecutive roots of f.

(ii) Because a and b are roots of f, $f(a) = f(b) = 0$. Moreover, f is continuous on $[a, b]$ and differentiable on (a, b) because it is a polynomial. Thus Rolle's Theorem implies that $f'(x) = 0$ for some x in (a, b).

Note that the derivative $f'(x) = (x - a)g(x) + (x - b)g(x) + (x - a)(x - b)g'(x)$, so that $f'(a) = (a - b)g(a) \ne 0$ and $f'(b) = (b - a)g(b) \ne 0$. □

Problem 4. (i) If $a_1 < a_2 < \cdots < a_n$, find the minimum value of $f(x) = \sum_{i=1}^{n}(x - a_i)^2$.

(ii) Find the minimum value of $f(x) = \sum_{i=1}^{n} |x - a_i|$.

(iii) (Not required) Let $a > 0$. Prove that the maximum value of

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - a|}$$

is $(2 + a)/(1 + a)$.

Solution. Done in class. □

Problem 5. (i) Suppose that $|f(x) - f(y)| \le |x - y|^\alpha$ for some $\alpha > 1$. Prove that f is constant.

(ii) Find a function f other than a constant function such that $|f(x) - f(y)| \le |x - y|$

Solution. (i) Becasue $\alpha > 1$, this inequality implies that $f'(x) = 0$ for all x:

$$|f'(x)| = \lim_{y \to 0} \frac{|f(x) - f(y)|}{|x - y|} \le \lim_{y \to 0} |x - y|^{\alpha - 1} = 0.$$

(ii) $f(x) = x/2$. □