Math 512A. Homework 9. Due 11/14/07

(Revised 11/10)

Problem 1. (i) Suppose that $g(x) = f(x + c)$ for all x. Prove, starting from the definition of the derivative, that $g'(x) = f'(x + c)$ for all x.

(ii) Prove that if $g(x) = f(cx)$, then $g'(x) = c \cdot f'(cx)$.

(iii) Suppose that f is differentiable and periodic, with period a, i.e., $f(x + a) = f(x)$ for all x. Prove that f' is also periodic with period a.

(iv) (Not required) Prove that if f is even, i.e., $f(x) = f(-x)$, then $f'(x) = -f'(-x)$.

(v) (Not required) Prove that if f is odd, i.e., $f(-x) = -f(x)$, then $f'(x) = f'(-x)$.

Problem 2. (i) Let $f(x) = x^2$ if x is rational, and $f(x) = 0$ if x is irrational. Prove that f is differentiable at 0.

(ii) Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that$f$ is differentiable at 0.

(iii) (Not required) Let $\alpha > 1$. Prove that if f satisfies $|f(x)| \leq |x|^\alpha$, then f is differentiable at 0.

Problem 3. Suppose that a and b are two consecutive roots of the polynomial function f, but that a and b are not double roots, so that we can write $f(x) = (x - a)(x - b)g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$.

(i) Prove that $g(a)$ and $g(b)$ have the same sign.

(ii) Prove that there is some number x with $a < x < b$ and $f'(x) = 0$.

(iii) (Not required) Prove that (ii) holds true even if a and b are multiple roots. Hint: If $f(x) = (x-a)^n(x-b)^m g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$, consider the polynomial function $h(x) = f'(x)/(x - a)^{n-1}(x - b)^{m-1}$.

Problem 4. (i) If $a_1 < a_2 < \cdots < a_n$, find the minimum value of $f(x) = \sum_{i=1}^{n}(x - a_i)^2$.

(ii) Find the minimum value of $f(x) = \sum_{i=1}^{n} |x - a_i|$.

(iii) (Not required) Let $a > 0$. Prove that the maximum value of

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - a|}$$

is $(2 + a)/(1 + a)$.

Problem 5. (i) Suppose that $|f(x) - f(y)| \leq |x - y|^\alpha$ for some $\alpha > 1$. Prove that f is constant.

(ii) Find a function f other than a constant function such that $|f(x) - f(y)| \leq |x - y|$