Math 512A. Homework 8 Solutions

Problem 1. Find the maximum value of $f(x) = x^3 - 9x$ in the interval $[-3, 3]$. **Note:** No derivatives!

Solution. This was done in class. □

Problem 2. Find an integer n such that the polynomial equation $x^3 - x + 3 = 0$ has a solution between n and $n + 1$.

Solution. We have $(-2)^3 - (-2) + 3 = -3$ and $(-1)^3 - (-1) + 3 = 1$, so there is a solution to the equation $x^3 - x + 3$ between $n = -2$ and $n + 1 = -1$. □

Problem 3. Prove that there is some number x such that $\sin x = x - 1$.

Solution. Let $f(x) = x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi/2 > 0$, so there is x in $(0, \pi/2)$ such that $f(x) = 0$, or $\sin x = x - 1$. □

Problem 4.

(i) Suppose that f is continuous on the interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for all x in $[0, 1]$. Prove that $f(x) = x$ for some number x in $[0, 1]$.

(ii) Let f be continuous and bounded above and below on $\mathbf{R}$. Prove that there is some number x such that $f(x) = x$.

Solution. If $f(0) = 0$ or if $f(1) = 1$, then we are done. If not, then $f(0) > 0$ and $f(1) < 1$. Let $g(x) = x - f(x)$. Then g is continuous on $[0, 1]$, $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. By the intermediate Value Theorem there is x in $[0, 1]$ such that $g(x) = 0$, or $f(x) = x$.

(ii) If f is bounded below and above on $\mathbf{R}$, then there are numbers a and b such that $a < f(x) < b$ for all x. The continuous function $g(x) = x - f(x)$ satisfies $g(a) < 0 < g(b)$. Apply the Intermediate Value Theorem to g on $[a, b]$. □

Problem 5. A function f defined on an interval I has the Intermediate Value Property on I if for any two numbers $a < b$ in I and every y strictly between $f(a)$ and $f(b)$, there is c in (a, b) such that $f(c) = y$.

(i) Prove that the function f given by $f(x) = \sin 1/x$ if $x \neq 0$ and $f(0) = 0$ has the Intermediate Value Property on the interval $[0, B]$, for any $B > 0$.

(ii) Prove that if f is non decreasing on the interval I and has the Intermediate Value Property on I, then f is continuous on I. (Recall that f is said to be non decreasing on I if $f(x) \leq f(y)$ whenever $x < y$ in I.)

Solution. (i) If $0 < a < b$ are two numbers in $[0, B]$, then we apply the Intermediate Value Theorem to $f(x) = \sin 1/x$ on the interval $[a, b]$ because f is continuous on $[a, b]$.

If $0 = a < b$ and x is strictly between $0 = f(0)$ and $f(b)$, let n be a natural number such that $2/b < (2n + 1)\pi$ so that the interval $J = [2/(2n + 3)\pi, 2/(2n + 1)\pi]$ is contained in $[0, b]$. The function $f(x) = \sin 1/x$ is continuous on J and takes on the values 1 and -1 at the endpoints of J. Since $-1 \leq f(x) \leq 1$, the Intermediate Value Theorem applied to f on J implies that given any y such that $-1 < y < 1$, there is c in J such that $f(c) = y$. In particular, if y is strictly between $f(a)$ and $f(b)$, then $-1 < y < 1$ also, and c in J satisfies $0 = a < c < b$, as desired.

(ii) Suppose that there is a in I where f fails to be continuous. Then there is a sequence (x_n) in I such that $x_n \to a$ but $f(x_n)$ does not converge to $f(a)$. We may assume, by taking a subsequence if necessary, that x_n increases (or decreases) to a. Then $f(x_n)$ is non decreasing and bounded above by $f(a)$, thus it converges to a number p with $p < f(a)$. Let q be a number such that $p < q < f(a)$. For each x_n we have $f(x_n) \leq p < q$, so the intermediate value property of f on the interval $[x_n, a]$ implies the existence of y_n in (x_n, a) such that $f(y_n) = q$. The sequence (y_n) converges to a and $f(y_n) = q$ for all n. Since x_n also converges to a, given n there is m such that $y_n < x_m$, but $f(y_n) = q > p \geq f(x_m)$, contradicting that f is non decreasing.

□