Math 512A. Homework 7. Due 10/31/07

Problem 1. Let $f : E \subset \mathbf{R} \to \mathbf{R}$ be uniformly continuous. Prove that if (x_n) is a Cauchy sequence in E, then $(f(x_n))$ is also a Cauchy sequence. Show by counterexample that *uniformly* is necessary.

Problem 2. (i) Prove that if f and g are uniformly continuous on E, then so is $f + g$.

(ii) Prove that if f and g are uniformly continuous and bounded on E, then fg is uniformly continuous on E.

(iii) Show that the conclusion in (ii) above does not hold if one of them is not bounded.

Problem 3. Let A and B be two nonempty sets of real numbers and suppose that $x \leq y$ for all x in A and all y in B.

(i) Prove that $\sup A \leq y$ for all y in B.

(ii) Prove that $\sup A \leq \inf B$.

   Note. The supremum of a set A, $\sup A$, was defined in Homework 6. The infimum of a set B is $\inf B = -\sup(-B)$, where $-B$ is the set of all numbers x such that $-x$ is in B.

Problem 4. (i) Consider a sequence of closed intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ... Suppose that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all n. Prove that there is a point x which is in every I_n.

(ii) Prove that if $\operatorname{length} I_n \to 0$, then the point x in (i) is unique.

(iii) Show that this conclusion in Part (i) is false if we consider open intervals instead of closed intervals. Is it true if we consider open and bounded intervals?

Problem 5. Suppose f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$.

(i) Prove that either $f((a + b)/2) = 0$, or f has different signs at the end points $[a, (a + b)/2]$, or f has different signs at the end points of $[(a + b)/2, b]$.

If $f((a+b)/2) \neq 0$, let I_1 be one of the two intervals on which f has different signs at the endpoints. Now bisect I_1. Then either f is 0 at the midpoint, of f has opposite signs at the endpoints of one of the two intervals into which I_1 was bisected. Let I_2 be such an interval. Continue in this way to define I_n for each natural number n (unless f is 0 at some midpoint).

(ii) Prove that there is a point x in (a, b) where $f(x) = 0$.

(iii) Use the scheme described in (i) and (ii) to approximate the solution of $x^3 + 6x - 2 = 0$ with an error smaller than $1/100$. (Calculators not allowed.)

Problem 6 (Not required). Let A and B be two nonempty sets of real numbers which are bounded above, and let $A + B$ denote the set of all real numbers of the form $x + y$ with x in A and y in B. Prove that $\sup(A + B) = \sup A + \sup B$.
Hint. The inequality $\sup(A + B) \leq \sup A + \sup B$ shoul dbe easy. To prove that $\sup A + \sup B \leq \sup(A + B)$, it suffices to prove that $\sup A + \sup B \leq \sup(A + B) + \varepsilon$ for all $\varepsilon > 0$. For this, begin by choosing x in A and y in B with $\sup A - x < \varepsilon/2$ and $\sup B - y < \varepsilon/2$.