Math 512A. Homework 6 Solutions

(Revised 10/27)

Problem 1. Prove the following:

(i) The intersection of an arbitrary family of compact sets is compact.

(ii) The union of finitely many compact sets is compact.

Solution. (i) Let $\{K_i\}_{i\in I}$ be a family of compact sets, and let $K = \bigcap_{i\in I} K_i$ denote their intersection. We'll show that K is compact by showing that it is closed and bounded. Each K_i is bounded (because it is compact) and $K \subset K_i$ (for all i) so K must be bounded, as any bound for K_i will also be a bound for K.

The set K is also closed because the intersection of closed sets is a closed set (Proposition 7.4)

(ii) Suppose that K_1 and K_2 are compact, and let $K = K_1 \cup K_2$ be their union. Let (x_n) be a sequence in K. Each x_n is in one of the two sets K_1 or K_2 (it could be in both), so it follows that there is a subsequence (x_{n_m}) of (x_n) where all the terms x_{n_m}, $m = 1, 2, \cdots$, are in the same K_i, $i = 1$ or 2. Since K_i is compact, this (sub)sequence (x_{n_m}) has a subsequence $(x_{n_{m_l}})$ which converges to a point x in K_i. But $(x_{n_{m_l}})$ is also a subsequence of the original sequence (x_n), and its limit x is in $K \supset K_i$.

This proves that the union of two compact sets is compact. For finite unions, the proof proceeds by induction on the number of sets. Suppose that you have proved that the union of $< n$ compact sets is a compact. If $K_1, \cdots, K_n$ is a collection of n compact sets, then their union can be written as $K = K_1 \cup (K_2 \cup \cdots \cup K_n)$, the union of two compact sets, hence compact. □

Problem 2. Prove or give a counterexample:

(i) The union of infinitely many compact sets is compact.

(ii) A non-empty subset S of real numbers which has both a largest and a smallest element is compact (*cf.* Proposition 8.3).

Solution. (i) False. We know that $\mathbf{R}$ is not compact (for example, because the sequence of natural numbers (n) has no convergent subsequence, or because $\mathbf{R}$ is not bounded), but we can write $\mathbf{R} = \bigcup_{n=1}^{\infty}[-n, n]$, where each interval $[-n, n]$ is compact because it is closed and bounded.

(ii) False. The set $S = [-1, 0) \cup (0, 1]$ has a largest element, namely 1, and a smallest element, namely -1, but it is not compact because, for example, the sequence $(1/n)$ is in S but has no subsequence which converges to a point in S. □

Problem 3. For a subset of real numbers S define the **supremum** $\sup S$ as follows:

$$\sup S = \begin{cases} \max \overline{S} & \text{if } S \text{ is nonempty and bounded above,} \\ +\infty & \text{if } S \text{ is nonempty and not bounded above,} \\ -\infty & \text{if } S \text{ is empty.} \end{cases}$$

Prove that, for S nonempty and bounded above, $\sup S \geq s$ for all s in S, and that $\sup S$ is the smallest number with that property.

Solution. Let S be nonempty and bounded above. Since $S \subset \overline{S}$, the greatest number in $\overline{S}$, which we called $\max \overline{S}$, is greater than any number in S. That is, $\sup S = \max \overline{S} \geq s$ for all s in S. Suppose that x is a number such that $x \leq \sup S$ and $x \geq s$ for all $s \in S$. We need to show that $x \geq \max \overline{S}$. Assume that this was not the case. Then there exists s_0 in $\overline{S}$ such that $x < s_0$. Because s_0 is in $\overline{S}$, there is a sequence (s_n) in S such that $s_n \to s$. For $\varepsilon = s_0 - x > 0$ there is a natural number N such that $|s_n - s_0| < \varepsilon$ if $n > N$; that is $x - s_0 < s_n - s_0 < s_0 - x$, for all $n > N$. But then

$$s_n - x = s_n - s_0 + s_0 - x > \frac{x - s_0}{2} + s_0 - x = \frac{s_0 - x}{2} > 0$$

for all $n > N$, contradicting that $x \geq s_n$ for all n. □

Problem 4. (i) Give an example of a continuous function on a closed set $E \subset \mathbf{R}$ that has no maximum.

(ii) Give an example of a continuous function on a bounded set $F \subset \mathbf{R}$ that has no maximum.

Solution. (i) The function $f(x) = x$ is continuous on the closed set $\mathbf{R}$, but it has no maximum.

(ii) The function $f(x) = 1/x$ is continuous on the bounded set $(0, 1)$ but has no maximum there. □

Problem 5. (i) Prove that the function $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

(ii) Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbf{R}$.

Solution. (i) Let $\varepsilon = 1$ and let $\delta > 0$ be arbitrary. W must find two numbers x and y in $(0, 1)$ such that $|x - y| < \delta$, but $|1/x - 1/y| \geq 1$.

If $\delta < 1$, let $x = \dfrac{-\delta + \sqrt{\delta^2 + 8\delta}}{4}$ and $y = \dfrac{x}{1-x}$. (**Note.** There are many other choices for x and y.) We easily verify that x and y are in $(0, 1)$. Indeed, for $0 < \delta < 1$,

$$\delta^2 < \delta^2 + 8\delta < \delta^2 + 4\delta + 4 = (\delta + 2)^2,$$

and taking square roots we have

$$\delta < \sqrt{\delta^2 + 8\delta} < \delta + 2$$

which implies

$$0 < x = \frac{-\delta + \sqrt{\delta^2 + 8\delta}}{4} < \frac{1}{2}.$$

If $0 < x < 1/2$, then $\dfrac{1}{2} < 1 - x < 1$, hence

$$0 < y = \frac{x}{1 - x} < 1.$$

We also have that

$$0 < y - x = \frac{x}{1 - x} - x = \frac{x^2}{1 - x} = \frac{\delta}{2} < \delta,$$

but

$$0 < \frac{1}{x} - \frac{1}{y} = 1.$$

If $\delta \geq 1$, take $x = 1/2$ and $y = 1/3$.

(ii) Let $\varepsilon = 1$ and let $\delta > 0$ be arbitrary. Take $x = 2/\delta$ and $y = 2/\delta + \delta/2$. Then $|x - y| = \delta/2 < \delta$ and $|x^2 - y^2| = 2 + (2/\delta)^2 > 1$. □