Math 512A. Homework 6. Due 10/24/07

Problem 1. Prove the following:

(i) The intersection of an arbitrary family of compact sets is compact.

(ii) The union of finitely many compact sets is compact.

Problem 2. Prove or give a counterexample:

(i) The union of infinitely many compact sets is compact.

(ii) A non-empty subset S of real numbers which has both a largest and a smallest element is compact (*cf.* Proposition 8.3).

Problem 3. For a subset of real numbers S define the **supremum** $\sup S$ as follows:

$$\sup S = \begin{cases} \max \overline{S} & \text{if } S \text{ is nonempty and bounded above,} \\ +\infty & \text{if } S \text{ is nonempty and not bounded above,} \\ -\infty & \text{if } S \text{ is empty.} \end{cases}$$

Prove that, for S nonempty and bounded above, $\sup S \geq s$ for all s in S, and that $\sup S$ is the smallest number with that property.

Problem 4. (i) Give an example of a continuous function on a closed set $E \subset \mathbf{R}$ that has no maximum.

(ii) Give an example of a continuous function on a bounded set $F \subset \mathbf{R}$ that has no maximum.

Problem 5. (i) Prove that the function $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

(ii) Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbf{R}$.