1 Math 512A. Homework 5 Solutions

Problem 1. (i) Define "countable set."

(ii) Determine (either prove or give a counterexample) whether the following statements are true: (a) The union of two uncountable sets is uncountable. (b) The intersection of two uncountable sets is uncountable.

Solution. (ii) (a) True. If $A \cup B$ is countable, then there is a one-one mapping $f : A \cup B \to \mathbf{N}$. The composite $f \circ i : A \to A \cup B \to \mathbf{N}$ is one-one, and hence A is countable. (b) False. $A = (-\infty, 1]$ and $B = [1, \infty)$ are uncountable but $A \cap B = \{1\}$ is countable. □

Problem 2. (i) Define "$\lim_{n\to\infty} a_n = l$."

(ii) Prove, using the definition in (i), that $a_n = \dfrac{2n-1}{n+3}$ converges to $l = 2$.

Solution. (ii) Given $\varepsilon > 0$, let $N = \max\{2, 7/\varepsilon - 2\}$. If $n > N$, then

$$\left|\frac{2n-1}{n+3} - 2\right| = \frac{7}{N+3} < \varepsilon.$$

□

Problem 3. Let a_n be the Fibonacci sequence, $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$.

(i) If $r_n = \dfrac{a_{n+1}}{a_n}$, then prove that $r_{n+1} = 1 + \dfrac{1}{r_n}$.

(ii) Prove that $r = \lim_{n\to\infty} r_n$ exists, and $r = 1 + \dfrac{1}{r}$. Conclude that $r = \dfrac{1+\sqrt{5}}{2}$.

Solution. (i) By definition, $a_{n+2} = a_{n+1} + a_n$. If $r_n = \dfrac{a_{n+1}}{a_n}$, then

$$r_{n+1} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1} + a_n}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} = 1 + \frac{1}{r_n}.$$

(ii) After looking at the first few terms of the sequence $(r_n) = (1, 2, 3/2, 5/3, 8/5, \ldots)$, we conjecture that the subsequence of (r_n) given by the terms with even indexes, (r_{2n}), is decreasing and that with odd indexes, (r_{2n-1}), is increasing. This is in fact true and can be proved by induction, using (i). To start, we have $r_1 < r_3$ and $r_4 < r_2$. If we know that $r_{2n} < r_{2n-2}$, then $r_{2n-1} = 1 + \dfrac{1}{r_{2n-2}} < 1 + \dfrac{1}{r_{2n}} = r_{2n+1}$, and similarly, if $r_{2n-1} < r_{2n+1}$, then $r_{2n+2} < r_{2n}$.

Furthermore, the sequences (r_{2n}) and (r_{2n-1}) are bounded because (r_n) is bounded on account of the identity proved in (*i*):

$$1 \le r_{n+1} = 1 + \frac{r_{n-1}}{r_{n-1}+1} \le 2.$$

Let $r_e = \lim_{n\to\infty} r_{2n}$ and $r_o = \lim_{n\to\infty} r_{2n-1}$. Then, by the properties of limits and the identities

$$r_{2n} = 1 + \frac{1}{r_{2n-1}} \qquad \text{and} \qquad r_{2n+1} = 1 + \frac{1}{r_{2n}}$$

we have

$$r_e = 1 + \frac{1}{r_o} \qquad \text{and} \qquad r_o = 1 + \frac{1}{r_e}$$

or $r_e r_o = r_e + 1 = r_o + 1$. Therefore, the limit $\lim r_n = r$ exists (Why?) and $r = r_e = r_o$ satisfies $r = 1 + \dfrac{1}{r}$, or $r^2 - r - 1 = 0$. Since r is positive, it must equal the positive solution of the quadratic equation $x^2 - x - 1 = 0$, that is, $r = \dfrac{1+\sqrt{5}}{2}$. □

Problem 4. (i) Find all the accumulation points of the set $S = \left\{ \frac{1}{n} + \frac{1}{m} \mid n \text{ and } m \text{ in } \mathbf{N} \right\}$

(ii) Prove that p is an accumulation point of a set $S \subset \mathbf{R}^n$ if and only if every ball about p contains infinitely many points of S.

Solution. (i) The accumulation points of S are $0, 1, 1/2, 1/3, \ldots$. Indeed, each of the numbers $\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{m}$, and $0 = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{n}$. In each case the point is limit of a sequence whose terms are all in S and are distinct from the point itself.

We prove that those are the only accumulation points of S. Suppose that $x \neq 0, 1, 1/2, \cdots$ is an accumulation point of S. Then there is a sequence (s_n) in S such that s_n converges to x and $s_n \neq x$ for all n. Each $s_n = 1/p_n + 1/q_n$ for some natural numbers p_n and q_n, with $p_n \leq q_n$. If the sequence (q_n) is bounded above, then so is the sequence p_n, and thus there are natural number p and q such that $p_n = q$ and $q_n = q$ for infinitely many n's. This implies that $s_n = 1/p + 1/q$ for infinitely many n's. Since a subsequence of a convergent sequence converges to the same limit, this forces $s_n = 1/p + 1/q = x$ for infinitely many n, which is a contradiction. Thus q_n has a subsequence q_{n_k} which is strictly increasing, hence such that $1/q_{n_k}$ converges to 0. If the corresponding subsequence of natural numbers (p_{n_k}) is bounded above, then it has a constant subsequence $p_{n_{k_l}} = p$ for some natural number p, which implies that $s_{n_{k_l}}$ converges to $1/p$, and thus that $x = 1/p$, again a contradiction. Therefore p_{n_k} has a strictly increasing subsequence $p_{n_{k_l}}$, which implies that $s_{n_{k_l}}$ and thus s_n, converges to 0, also a contradiction. □

Problem 5. (i) Let a_n be a bounded injective sequence of real numbers. Prove that if p is the only accumulation point of the set $A = \{a_n \mid n \text{ in } \mathbf{N}\}$, then the sequence a_n converges and $\lim_{n\to\infty} a_n = p$.

(ii) Show by a counterexample that this property is not true for unbounded sequences.

Solution. (i) Suppose that a_n does not converge to p. Then there is $\varepsilon > 0$ such that for every natural number k, there is $n_k > k$ such that $|a_{n_k} - p| \geq \varepsilon$. The sequence (a_{n_k}) is a subsequence of (a_n) and thus it is bounded. Therefore it has a subsequence $(a_{n_{k_l}})$ which converges to a point $q \neq p$ (because $|a_{n_{k_l}} - p| > \varepsilon$). Because the sequence (a_n) is injective, all elements of this subsequence are distinct and therefore q is an accumulation point of the set A.

Note. If "injective" is not assumed, then (i) may not be true. Let $a_n = 1$ is n is odd and $a_n = 1/n$ if n is even. Then the set $A = \{a_n\} = \{1, 1/2, 1/4, 1/6, \cdots\}$ has only one accumulation point, namely 0, but the original sequence a_n does not converge to 0 (or to any other number).

(ii) Let $a_n = n$ if n is odd and $a_n = 1/n$ if n is even. The sequence (a_n) is unbounded and 0 is the only accumulation point of the set $A = \{a_n\} = \{1, 1/2, 3, 1/4, 5, 1/6, \cdots\}$. □

Problem 6. (i) Define the concept "bounded sequence."

(ii) Prove that a set $S \subset \mathbf{R}$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Solution. (i) A sequence (x_n) is bounded if there is a number M such that $|x_n| \leq M$ for all n.

(ii) Assume that $S \subset \mathbf{R}$ is bounded. If (x_n) is a sequence in S, then (x_n) is bounded and thus it has a convergent subsequence.

Assume that S is not bounded. Then for any integer n there is an x_n in S such that $|x_n| > n$. Since any convergent sequence is bounded, the sequence (x_n) cannot have a convergent subsequence. □

Problem 7. (i) Define the concept "f is a continuous function at the point p."

(ii) Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = x$ if x is rational, and $f(x) = -x$ if x is irrational. Prove that f is continuous only at $p = 0$.

Solution. (i) f is continuous at p if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < |x - p| < \delta$ and x is in the domain of f, then $|f(x) - f(p)| < \varepsilon$.

The following equivalent version of continuity will be used bellow: f if continuous at p if and only if $f(x_n) \to f(p)$ for any sequence (x_n) in domain f such that $x_n \to p$.

(ii) f is continuous at 0. Given $\varepsilon > 0$, take $\delta = \varepsilon$. If $|x| < \delta$, then

$$|f(x) - f(0)| = |f(x)| = |x| < \delta = \varepsilon.$$

If $p \neq 0$, then f is not continuous at p. Indeed, if p is not rational there is a sequence p_n of rational numbers such that $p_n \to p$, and if p is rational, there is a sequence of irrational numbers $p_n \to p$ (for example, $p_n = p + \sqrt{2}/n$). In either case, $\lim_{n\to\infty} f(p_n) = -p$, which is $\neq p$ for $p \neq 0$. □

Problem 8. (i) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, can $\lim_{x\to a}[f(x)+g(x)]$ or $\lim_{x\to a}(f\cdot g)(x)$ exist?

(ii) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}[f(x)+g(x)]$ exists, must $\lim_{x\to a} g(x)$ exist?

(iii) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a}[f(x)+g(x)]$ exist?

(iv) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

Solution. (i) Yes. Let $f(x) = 1$ if x is rational and $f(x) = -1$ if x is not rational, and let $g = -f$. Then $f(x) + g(x) = 0$ for all x, and $f(x)g(x) = -1$ for all x.

(ii) Yes. Write $g(x) = (f(x) + g(x)) - f(x)$. Each of the terms $f(x) + g(x)$ and $f(x)$ on the right side has limit when $x \to a$, so their difference also has limit when $x \to a$.

(iii) No. Apply (ii).

(iv) No. Let $f(x) = 0$ for all x and let $g(x) = 1$ if x is rational and $g(x) = -1$ if x is not rational. Then, for any a, $f(x)$ and $f(x)g(x) = 0$ both have limit when $x \to a$, but $g(x)$ does not have limit when $x \to a$. □

Problem 9. (i) Define the concepts "closed set" and "closure of a set."

(ii) Prove that the closure of a set $S \subset \mathbf{R}^n$ is the smallest closed subset of $\mathbf{R}^n$ which contains S.

Solution. (i) A set is closed if it contains all its boundary points. The closure of a set S is $\overline{S} = S \cup \partial S$.

(ii) By definition, the closure of S is $\overline{S} = S \cup \partial S$, hence $\overline{S}$ contains S.

We show that $\overline{S}$ is closed. Let p be point in of $\partial\overline{S}$, the boundary of $\overline{S}$. Then every ball around p intersects $\overline{S}$ and its complement $\overline{S}^c$. Hence there is a sequence (x_n) in $\overline{S}$ such that $|x_n - p| < 1/n$ for all n. Because x_n is in $\overline{S}$, there is y_n in S such that $|y_n - x_n| < 1/n$. Then, by the triangle inequality, $|y_n - p| \le |y_n - x_n| + |x_n - p| < 2/n$. Thus p is in $\overline{S}$ because the (y_n) is in S and converges to p.

We show that $\overline{S}$ is the smallest closed set containing S, that is, if T is closed and $S \subset T$, then $\overline{S} \subset T$. Assume this was not the case. Then there is p in $\overline{S}$ which is also in T^c. In particular, p is not in S. Moreover, since T^c is open, there is a ball around p which is completely contained in T^c. This ball cannot intersect S because $S \subset T$, and this p cannot be a boundary point of S. Hence p is not in $S \cup \partial S = \overline{S}$, a contradiction. □