Math 512A. Homework 5. Due 10/10/07

(Work out any 5 problems)

(Revised 10/4)

Problem 1. (i) Define "countable set."

(ii) Determine (either prove or give a counterexample) whether the following statements are true: (a) The union of two uncountable sets is uncountable. (b) The intersection of two uncountable sets is uncountable.

Problem 2. (i) Define "$\lim_{n\to\infty} a_n = l$."

(ii) Prove, using the definition in (i), that $a_n = \dfrac{2n-1}{n+3}$ converges to $l = 2$.

Problem 3. Let a_n be the Fibonacci sequence, $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$.

(i) If $r_n = \dfrac{a_{n+1}}{a_n}$, then prove that $r_{n+1} = 1 + \dfrac{1}{r_n}$.

(ii) Prove that $r = \lim_{n\to\infty} r_n$ exists, and $r = 1 + \dfrac{1}{r}$. Conclude that $r = \dfrac{1+\sqrt{5}}{2}$.

Problem 4. (i) Find all the accumulation points of the set $\left\{\dfrac{1}{n} + \dfrac{1}{m} \mid n \text{ and } m \text{ in } \mathbf{N}\right\}$

(ii) Prove that p is an accumulation point of a set $S \subset \mathbf{R}^n$ if and only if every ball about p contains infinitely many points of S.

Problem 5. (i) Let a_n be a bounded sequence of real numbers. Prove that if p is the only accumulation point of the set $A = \{a_n \mid n \text{ in } \mathbf{N}\}$, then the sequence a_n converges and $\lim_{n\to\infty} a_n = p$.

(ii) Show by a counterexample that this property is not true for unbounded sequences.

Problem 6. (i) Define the concept "bounded sequence."

(ii) Prove that a set $S \subset \mathbf{R}$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Problem 7. (i) Define the concept "f is a continuous function at the point p."

(ii) Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = x$ if x is rational, and $f(x) = -x$ if x is irrational. Prove that f is continuous only at $p = 0$.

Problem 8. (i) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, can $\lim_{x\to a} \big[f(x) + g(x)\big]$ or $\lim_{x\to a} (f \cdot g)(x)$ exist?

(ii) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} \big[f(x) + g(x)\big]$ exists, must $\lim_{x\to a} g(x)$ exist?

(iii) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a} \big[f(x) + g(x)\big]$ exist?

(iv) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

Problem 9. (i) Define the concepts "closed set" and "closure of a set."

(ii) Prove that the closure of a set $S \subset \mathbf{R}^n$ is the smallest closed subset of $\mathbf{R}^n$ which contains S.