Math 512A. Homework 4. Due 10/3/07

(Revised 9/30)

Problem 1. (You must show that your example satisfies the required property.)

(i) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is continuous except at the integers.

(ii) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is continuous only at 0.

(iii) For each number p, find a function which is continuous at p, but not at any other points.

(iv) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is discontinuous at 1, 1/2, 1/3, ... but continuous at all other points.

(v) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is discontinuous at 1, 1/2, 1/3, ..., and at 0, but continuous at all other points.

Problem 2. (i) Prove that if f is continuous at p, then so is $|f|$.

(ii) Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere.

(iii) Suppose that g is continuous at 0, $g(0) = 0$, and f is a function satisfying $|f(x)| \leq |g(x)|$ for all x. Show that f is continuous at 0.

(iv) Prove that if f and g are continuous, so are $\max\{f, g\}$ and $\min\{f, g\}$

Problem 3. (i) Suppose that $\lim\limits_{x\to 0} f(x)$ exists. Prove that if g is any function such that $\lim\limits_{x\to 0} g(x)$ does not exists, then $\lim\limits_{x\to 0} [f(x) + g(x)]$ also does not exist.

(ii) Suppose that $\lim\limits_{x\to 0} f(x)$ exists and is $\neq 0$. Prove that if g is any function such that $\lim\limits_{x\to 0} g(x)$ does not exists, then $\lim\limits_{x\to 0} [f(x) \cdot g(x)]$ also does not exist.

Problem 4. (i) Prove that if f and g are continuous, then so is $f + g$.

(ii) Prove that if f and g are continuous, then so is $f \cdot g$.

(iii) Give an example of two functions f and g, both discontinuous at 0, whose sum is continuous at 0.

(iv) Give an example of two functions f and g, both discontinuous at 0, whose product is continuous at 0.

Problem 5. (i) Suppose that $\lim\limits_{x\to p} f(x) = a \neq 0$. Prove that $\lim\limits_{x\to p} \frac{1}{f}(x) = \frac{1}{a}$.

(ii) Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomials in x. Prove that r is continuous at all x such that $q(x) \neq 0$.

Problem 6 (Not required)**.** Let (a_n) be a sequence of real numbers such that $0 < a_1 < 1$, and for all $n > 1$, $a_{n+1} = \frac{2}{1 + a_n}$.

(i) Prove that the subsequences (a_{2n}) and (a_{2n+1}) are both monotonic, one decreasing and the other increasing.

(ii) Prove that (a_n) converges, and find its limit.