Math 512A. Homework 3. Solutions

Problem 1. Find all the accumulation points of the following sets:

(i) The interval $[0, 1)$.

(ii) The set of all the irrational numbers.

(iii) The set of the natural numbers.

Solution. (i) The interval $[0, 1]$. (ii) The set of all real numbers. (iii) The empty set. □

Problem 2. A sequence (a_n) is said to be Cauchy if, for every $\varepsilon > 0$, there is a natural number N such that whenever $n, m > N$, $|a_n - a_m| < \varepsilon$.

(i) Prove that a convergent sequence of real numbers is Cauchy.

(ii) Prove that a Cauchy sequence is bounded.

Proof. Solution(i) Suppose that $a_n \to l$. Given $\varepsilon > 0$ there is a natural number N such that if $n > N$, then $|a_n - l| < \varepsilon/2$. Therefore, if $p, q > N$,

$$|a_p - a_q| = |a_p - l + l - a_q| \leq |a_p - l| + |a_q - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus showing that (a_n) is Cauchy. □

Problem 3. Prove or give a counterexample:

(i) If (a_n) is an increasing sequence (that is, $a_1 < a_2 < a_3 < \cdots$) such that $\lim_{n\to\infty} \big(a_{n+1} - a_n\big) = 0$, then (a_n) is convergent.

(ii) If (a_n) is increasing and bounded above, and $\lim_{n\to\infty} a_n = l$, then $a_n \leq l$.

Proof. Solution(i) Let $a_n = \sqrt{n}$. Then $a_n < a_{n+1}$ and (rationalizing)

$$0 < a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{(n+1)n}} \leq \frac{1}{n+1}$$

which implies that $\lim_{n\to\infty} (a_{n+1} - a_n) = 0$.
(ii) If there is a natural number N such that $a_N > l$, then $a_n \geq a_N > l$ for all natural numbers $n \geq N$, and a_n cannot converge to l. □

Problem 4. (i) Give an example of a sequence of real numbers with subsequences converging to every integer.

(ii) Give an example of a sequence of real numbers with subsequences converging to every real number.

Proof. Solution (i) 0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3 ..., and so on. Each integer appear infinitely many times in this sequence and thus you can extract a subsequence which converges to any integer (in fact, a constant sequence). □

Problem 5. Prove that if the subsequences (a_{2n}) and (a_{2n+1}) of a sequence (a_n) of real numbers both converge to the same limit l, then (a_n) converges to l.

Proof. Solution Given $\varepsilon > 0$ there are natural numbers N_e and N_o such that if n is an even natural number and $n > N_e$, then $|a_n - l| < \varepsilon$, and if n is an odd natural number and $n > N_o$, then $|a_n - l| < \varepsilon$. Let $N = \max\{N_e, N_o\}$. If n is a natural number and $n > N$, then n is either even and $> N_e$, or n is odd and $> N_o$. In either case, $|a_n - l| < \varepsilon$. □