Math 512A. Homework 3. Due 9/19/07

Problem 1. Find all the accumulation points of the following sets:

(i) The interval $[0, 1)$.

(ii) The set of all the irrational numbers.

(iii) The set of the natural numbers.

Problem 2. A sequence (a_n) is said to be Cauchy if, for every $\varepsilon > 0$, there is a natural number N such that whenever $n, m > N$, $|a_n - a_m| < \varepsilon$.

(i) Prove that a convergent sequence of real numbers is Cauchy.

(ii) Prove that a Cauchy sequence is bounded.

Problem 3. Prove or give a counterexample:

(i) If (a_n) is an increasing sequence (that is, $a_1 < a_2 < a_3 < \cdots$) such that $\lim_{n\to\infty} \big(a_{n+1} - a_n\big) = 0$, then (a_n) is convergent.

(ii) If (a_n) is increasing and bounded above, and $\lim_{n\to\infty} a_n = l$, then $a_n \leq l$.

Problem 4. (i) Give an example of a sequence of real numbers with subsequences converging to every integer.

(ii) Give an example of a sequence of real numbers with subsequences converging to every real number.

Problem 5. Prove that if the subsequences (a_{2n}) and (a_{2n+1}) of a sequence (a_n) of real numbers both converge to the same limit l, then (a_n) converges to l.