Math 512A. Homework 2. Solutions

Problem 1. Recall that the absolute value $|a|$ of a real number a is given by

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0. \end{cases}$$

Prove the following:

(i) $|a+b| \leq |a| + |b|$.

(ii) $|a-b| \leq |a| + |b|$.

(iii) $|a| - |b| \leq |a-b|$.

(iv) $\big||a| - |b|\big| \leq |a-b|$.

Solution. (i) We consider 4 cases:

$$\begin{array}{lll} (1) & a \geq 0, & b \geq 0 \\ (2) & a \geq 0, & b \leq 0 \\ (3) & a \leq 0, & b \geq 0 \\ (4) & a \geq 0, & b \leq 0 \end{array}$$

In case (1) we have $|a| = a$ and $|b| = b$, and we also have $a+b \geq 0$, so $|a+b| = a+b$, making the result obvious: $|a+b| = a+b = |a| + |b|$, so that in fact we have equality. Case (4) is similar to case (1). Indeed we have $|a| = -a$, $|b| = -b$ and also $|a+b| = -(a+b) = -a-b$ because $a+b| \leq 0$.

In case (2) we have $a \geq 0$ and $b \leq 0$, hence $|a| = a$ and $|b| = -b$, so we must prove that

$$|a+b| \leq a - b.$$

We divide the proof into two subcases

$$\begin{array}{ll} (2a) & a+b \leq 0, \\ (2b) & a+b \geq 0. \end{array}$$

If case (2a) holds, then $|a+b| = -(a+b) = -a-b$, and we must show that $-a-b \leq a-b$, or that $-a \leq a$. This is certainly true because $a \geq 0$ implies that $-a \leq 0 \leq a$.

If case (2b) holds, then $|a+b| = a+b$, and we must show that $a+b \leq a-b$, or that $b \leq -b$. But the hypothesis for case (2) is that $b \leq 0$, so that $-b \geq 0$ and hence $-b \leq 0 \leq b$.

Case (3) requires no additional work; it follows by applying case (2) with a and b interchanged.

(ii) By (i),

$$|a-b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

(iii) By (i),

$$|a| = |a-b+b| \leq |a-b| + |b|$$

and subtracting $|b|$

$$|a| - |b| \leq |a-b|$$

(iv) By (iii) we have that $|a|-|b| \leq |a-b|$, and by interchanging a and b we also have that $|b|-|a| \leq |b-a| = |a-b|$. Therefore,

$$\big||a|-|b|\big| = \begin{cases} |a|-|b| \leq |a-b| & \text{if } |a|-|b| \geq 0, \\ |b|-|a| \leq |a-b| & \text{if } |a|-|b| \leq 0. \end{cases}$$

□

Problem 2. Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove the following:

(i) $a_n + b_n \to a + b$.

(ii) $a_n \cdot b_n \to a \cdot b$.

Solution. (ii) By adding and subtracting $a_n b$ and then rearranging terms, we obtain

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

Because the sequences a_n and b_n both converge, they are both bounded: there is $M > 0$ such that $|a_n| \leq M$ and $|b_n| \leq M$ for all n. Therefore:

$$|a_n b_n - ab| \leq M|a_n - a| + M|b_n - b|,$$

for all natural numbers n.

Let $\varepsilon > 0$. There is a natural number N_1 such that if $n > N_1$, then $|a_n - a| < \varepsilon/2M$, and there is a natural number N_2 such that if $n > N_2$, then $|b_n - b| < \varepsilon/2M$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then

$$|a_n b_n - ab| \leq M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon.$$

□

Problem 3. (i) Prove that if $a_n \leq b_n$, if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $a \leq b$.

(ii) Prove that if $a_n \leq c_n \leq b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, then $\lim_{n\to\infty} c_n = l$.

Solution. (i) Suppose that $a > b$ and apply the definition of limit to $\varepsilon = \dfrac{a-b}{2}$. There is a natural number N_1 such that if $n > N_1$, then $|a_n - a| < \dfrac{a-b}{2}$ and a natural number N_2 such that if $n > N_2$, then $|b_n - b| < \dfrac{a-b}{2}$. If $n > \max\{N_1, N_2\}$, then

$$b_n < \frac{a-b}{2} + b = a - \frac{a-b}{2} < a_n,$$

which contradicts the hypothesis. □

Problem 4. Verify the following limits

(i) $\displaystyle\lim_{n\to\infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$.

(ii) $\displaystyle\lim_{n\to\infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} = \frac{1}{2}$.

(iii) $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (Hint: put $\sqrt[n]{n} = 1 + a_n$, prove that $a_n > 0$ for $n > 1$, deduce that $n - 1 \geq \frac{1}{2}n(n-1)a_n^2$ for $n > 1$, hence that $0 \leq a_n^2 \leq 2/n$.)

Solution. (i) Divide numerator and denominator by n^3 and obtain:

$$\frac{3n^3+7n^2+1}{4n^3-8n+63} = \frac{3+\dfrac{7}{n}+\dfrac{1}{n^3}}{4-\dfrac{8}{n^2}+\dfrac{63}{n^3}}.$$

The limit of the numerator is (by using the algebraic properties of limits):

$$\lim_{n\to\infty} 3+\frac{7}{n}+\frac{1}{n^3} = \lim_{n\to\infty} 3 + 7\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty} 1n^3 = 3$$

and by similar arguments, the limit of the denominator is

$$\lim_{n\to\infty} 4-\frac{8}{n^2}+\frac{63}{n^3} = 4$$

Since this limit is $\neq 0$, we have

$$\lim_{n\to\infty}\frac{3n^3+7n^2+1}{4n^3-8n+63} = \frac{3}{4}$$

(ii) Divide numerator and denominator by 2^{n+1}.
(iii) Let $\sqrt[n]{n} = 1 + a_n$. It is clear that $a_n \geq 0$, for if $a_n < 0$, then $n = (1+a_n)^n < 1$. By the binomial theorem

$$\begin{aligned} n = (1+a_n)^n &= 1 + na_n + \frac{n(n-1)}{2}a_n^2 + \cdots \\ &\geq 1 + \frac{n(n-1)}{2}a_n^2 \qquad \text{(we removed positive terms form the sum above)}, \end{aligned}$$

which implies that

$$0 \leq a_n \leq \frac{\sqrt{2}}{\sqrt{n}}$$

and thus that It follows from Problem 3(ii) that $\lim_{n\to\infty} a_n = 0$, and hence that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. □

Problem 5. Does the sequence converge or diverge? If it converges, what is the limit?

(i) $a_n = \dfrac{n}{n+1} - \dfrac{n+1}{n}$.

(ii) $a_n = \dfrac{2^n}{n!}$.

(iii) $a_n =$ the nth decimal digit of π (thus $a_1 = 1$, $a_2 = 4$, $a_3 = 1$, and so on).

Solution. (ii) We have the inequalities

$$0 \leq \frac{2^n}{n!} = \frac{2\cdot 2\cdot \ldots \cdot 2}{n(n-1)\cdot\ldots\cdot 2\cdot 1} \leq \frac{4}{n}$$

so Problem 3(ii) implies that $\lim_{n\to\infty}\dfrac{2^n}{n!} = 0$.
(iii) This assumes that you know that π is not a rational number, thus that its decimal digit expansion does not eventually repeat.

Let a_n be the nth decimal digit of π, and suppose that (a_n) converges, say $\lim_{n\to\infty} a_n = a$. Because all a_n are decimal digits, a must also be a decimal digit. Indeed, if not, $|a_n - a| \geq \min\{|a-d| \mid d = 0,1,2,3,4,5,6,7,8,9\} > 0$, contradicting $a_n \to a$. In fact, we must have $a_n = a$ eventually. Indeed, for $\varepsilon = 1$ there is a natural number N such that if $n > N$, then $|a_n - a| < 1$, hence that $a_n = a$ because two decimal digits either are equal or their difference is at least 1 in absolute value. The fact that $a_n = a$ eventually implies that π is a rational number, a contradiction. □