# Math 512A. Homework 2. Due 9/12/07

**Problem 1.** Recall that the absolute value $|a|$ of a real number $a$ is given by

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0. \end{cases}$$

Prove the following:

(i) $|a+b| \leq |a| + |b|$.

(ii) $|a-b| \leq |a| + |b|$.

(iii) $|a| - |b| \leq |a-b|$.

(iv) $\big||a| - |b|\big| \leq |a-b|$.

**Problem 2.** Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove the following:

(i) $a_n + b_n \to a + b$.

(ii) $a_n \cdot b_n \to a \cdot b$.

**Problem 3.** (i) Prove that if $a_n \leq b_n$, if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $a \leq b$.

(ii) Prove that if $a_n \leq c_n \leq b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, then $\lim_{n\to\infty} c_n = l$.

**Problem 4.** Verify the following limits

(i) $\displaystyle\lim_{n\to\infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$.

(ii) $\displaystyle\lim_{n\to\infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} = \frac{1}{2}$.

(iii) $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (Hint: put $\sqrt[n]{n} = 1 + a_n$, prove that $a_n > 0$ for $n > 1$, deduce that $n - 1 \geq \frac{1}{2}n(n-1)a_n^2$ for $n > 1$, hence that $0 \leq a_n^2 \leq 2/n$.)

**Problem 5.** Does the sequence converge or diverge? If it converges, what is the limit?

(i) $\displaystyle a_n = \frac{n}{n+1} - \frac{n+1}{n}$.

(ii) $\displaystyle a_n = \frac{2^n}{n!}$.

(iii) $a_n =$ the $n$th decimal digit of $\pi$ (thus $a_1 = 1$, $a_2 = 4$, $a_3 = 1$, and so on).