Math 512A. Homework 10. Solutions

Problem 1. Prove that if $f(x) = x^3$, then $\int_0^b f = \frac{b^4}{4}$, by considering upper and lower sums for partitions of $[0, b]$ into n equal subintervals, using the formula $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for the sum of the cubes of the first n natural numbers.

Problem 2. Decide which of the following functions are integrable on $[0, 2]$, and calculate the integral $\int_0^2 f$ if you can.

(i) $f(x) = \begin{cases} x + [x], & x \text{ rational} \\ 0, & x \text{ not rational.} \end{cases}$

(ii) f is the function whose graph is depicted in the figure below (set $f(0) = 0$).

Solution. (i) If P is any partition of $[0, 2]$, then (clearly) $L(f, P) = 0$ and also $U(f, P) \geq 1$. To see why this inequality holds, let $P = \{t_0, t_1, \cdots, t_n\}$ and let t_k be first element in P such that $t_k \geq 1$. Then $M_i \geq 1$ for $i \geq k$, and so, since $f \geq 0$,

$$U(f, P) \geq \sum_{i=k}^{n} (t_i - t_{i-1}) = 2 - t_{k-1} \geq 1.$$

Therefore $\sup_P L(f, P) = 0 < 1 \leq \inf_P U(f, P)$ and thus f can not be integrable on $[0, 2]$.

(ii) Discussed in class. Please refer also to the hints posted on the 512A webpage. □

Problem 3. (i) Prove that if f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f \geq 0$.

(ii) Prove that if f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then $\int_a^b f \geq \int_a^b g$. (Warning: If you work hard on part (b), then you are waisting time.)

(iii) Give an example of an f which is integrable on $[a, b]$, satisfies $f(x) \geq 0$ for all x, and $f(x) > 0$ for some x, and yet $\int_a^b f = 0$.

(iv) Suppose that $f(x) \geq 0$ for all x in $[a, b]$ and f is continuous at x_0 in $[a, b]$ and $f(x_0) > 0$. Prove that $\int_a^b f > 0$. (Hint. It suffices to find a partition P for which the lower sum $L(f, P) > 0$.

Solution. (i) If $P = \{t_0, t_1, \cdots, t_n\}$ is a partition of $[a, b]$, then $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \geq 0$ and so $L(f, P) \geq 0$. Therefore

$$\int_a^b f = \sup_P L(f, P) \geq 0.$$

(ii) If $f \geq g$ are integrable, then $f - g$ is integrable. By (i), $\int_a^b (f - g) \geq 0$. But $\int_a^b (f - g) = \int_a^b f - \int_a^b g$.

(iii) Take $f(x) = 0$ if $a < x < b$ and $f(a) = f(b) = 1$.

(iv) If $f \geq 0$ is continuous and $f(x_0) = y_0 > 0$, then there is $\delta > 0$ such that $f(x) > y_0/2$ for all x in $[x_0 - \delta, x_0 + \delta]$. If P is the partition $P = \{a, x_0 - \delta, x_0 + \delta, b\}$, then $L(f, P) \geq \dfrac{y_0 \delta}{2} > 0$, so $\int_a^b f \geq \dfrac{y_0 \delta}{2}$.

□

Problem 4. Suppose that f and g are integrable on $[a, b]$. If P is a partition of $[a, b]$, let M_i' and m_i' the appropriate sup's and inf's for f on the intervals of P, define M_i'' and m_i'' similarly for g, and define M_i and m_i similarly for the product fg.

Assume that $f(x) \geq 0$ and $g(x) \geq 0$ for all x in $[a, b]$.

(i) Prove that $M_i \leq M_i' M_i''$ and $m_i \geq m_i' m_i''$.

(ii) Prove that

$$U(P, fg) - L(P, fg) \leq \sum_{i=1}^{n} \left(M_i' M_i'' - m_i' m_i''\right)(t_i - t_{i-1}).$$

(iii) Use the fact that f and g are bounded (so that $|f(x)| \leq M$ and $|g(x)| \leq M$, for all x in $[a, b]$, to prove that

$$U(P, fg) - L(P, fg) \leq M \left\{U(P, f) + U(P, g) - L(P, f) - L(P, g)\right\}$$

(iv) Prove that fg is integrable.

(v) (Not required) Remove the condition that $f(x) \geq 0$ and $g(x) \geq 0$ on $[a, b]$.

Solution. Discussed in class. Please refer also to the hints posted on the 512A webpage. □

Problem 5. (i) (Schwarz Inequality) Prove that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 \qquad (*)$$

for real numbers $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$. There are many proofs available; one of them starts by first establishing the identity

$$\sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 = \left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{i<j} (x_i y_j - x_j y_i)^2.$$

(ii) Prove that equality in $(*)$ holds if and only if there is a real number λ such that $x_i = \lambda y_i$ for all $i = 1, \cdots, n$.

(iii) (Cauchy-Schwarz inequality) Suppose that f and g are integrable on $[a, b]$. Prove that

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right). \qquad (**)$$

(iv) If equality holds in $(**)$, is it necessarily true that $f = \lambda g$ for some real number λ? What if f and g are continuous?

Solution. Discussed in class. Please refer also to the hints posted on the 512A webpage. □