Math 512A. Homework 10. Hints

Problem 2 (ii) This f is not continuous on $[0,2]$ (why?) but it is continuous on $[a,2]$ for any $a>0$, hence integrable there. Given $\varepsilon>0$, use that to find a partition $P=\{t_1=\varepsilon/2, t_2, \cdots, t_n\}$ of $[\varepsilon/2, 2]$ for which $U(P,f)-L(P,f)<\varepsilon/2$ (on $[\varepsilon/2,2]$) and then look at the partition $\{0, \varepsilon/2, t_1, \cdots, t_n\}$ of $[0,2]$.

Of course, you still need to find the value $\int_0^2 f$. You can further elaborate the same idea.

Slightly more generally, if you know that an f is integrable on $[a,b]$, then you know that for each n there is a partition P_n of $[a,b]$ for which $U(P_n,f)-L(P_n,f)<1/n$, and hence that

$$\lim_n U(P_n,f) = \lim_n L(P_n,f) = \int_a^b f$$

because

$$L(P_n,f) \le \int_a^b f \le U(P_n,f).$$

Problem 5 (iii) If $P=\{t_0, t_1, \cdots, t_n\}$ is a partition of $[a,b]$, then, with the notation of Problem 4,

$$\begin{aligned} U(P,fg)^2 &= \left(\sum_{i=1}^n M_i(t_i-t_{i-1})\right)^2 \\ &\le \left(\sum_{i=1}^n M_i' M_i''(t_i-t_{i-1})\right)^2 \\ &\le \left(\sum_{i=1}^n \left[M_i'\sqrt{t_i-t_{i-1}}\right]\left[M_i''\sqrt{t_i-t_{i-1}}\right]\right)^2 \end{aligned}$$

Now apply the Schwarz inequality from Part (i), etcetera.