Homework 12. Solutions

¶ **1.** Let $A \subset X$ be a retract, let $r : X \to A$ be the retraction mapping, and let $j : A \to X$ be the inclusion mapping. Prove that, for any a in A, the homomorphism $r_* : \pi_1(X; a) \to \pi_1(A; a)$ is surjective, and the homomorphism $j_* : \pi_1(A; a) \to \pi_1(X; a)$ is injective.

Solution. The composition $r \circ j : A \to A$ is the identity mapping of A: $r \circ j = \mathrm{id}_A$. Therefore, $(r \circ j)_* = r_* \circ j_*$ is the identity homomorphism of $\pi_1(A; a)$.
If $z \in \pi_1(A; a)$, then $j_* z \in \pi_1(X; a)$ and $r_*(j_* z) = z$, showing that r_* is surjective.
If $z \in \pi_1(A; a)$ is such that $j_* z$ is the identity element of $\pi_1(X; a)$, then $r_* j_* z = z$ must be the identity element of $\pi_1(A; a)$, showing that j_* is injective. □

¶ **2.** Prove that the unit circle $S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ is not a retract of the plane $\mathbf{R}^2$.

Solution. By Problem 1, a retraction $r : \mathbf{R}^2 \to S^1$ induces a surjective homomorphism $r_* : \pi_1(\mathbf{R}^2; 1) \to \pi_1(S^1; 1)$. This is not possible because $\mathbf{R}^2$ is simply connected, hence $\pi_1(\mathbf{R}^2; 1)$ is the trivial group, while $\pi_1(S^1; 1)$ is infinite cyclic. □

¶ **3.** Prove that for a path connected space X the following are equivalent:

(a) X is simply connected;

(b) every continuous mapping $f : S^1 \to X$ is homotopic to a constant mapping.

(c) every continuous mapping $f : S^1 \to X$ admits a continuous extension to the closed disk B^2;

(d) any two paths $\alpha, \beta : [0, 1] \to X$ with the same initial and final points are homotopic relative to endpoints.

¶ **4.** If X is a topological space and γ is a path in X from $\gamma(0) = a$ to $\gamma(1) = b$, then the mapping $\gamma_* : \pi_1(X; a) \to \pi_1(X; b)$ given by $\gamma_*[\alpha] = [\gamma^{-1}\alpha\gamma]$ is an isomorphism.
Prove that $\gamma_* = \gamma'_*$ for any paths $\gamma, \gamma' \in \prod(X; a, b)$ if and only if $\pi_1(X, a)$ is commutative.

¶ **5.** Let (X, a) and (Y, b) be pointed spaces. Prove that the fundamental group $\pi_1(X \times Y; (a, b))$ is isomorphic to the direct product $\pi_1(X; a) \times \pi_1(Y; b)$.

Solution. Textbook, page 222. □

¶ **6.** Prove that if there are two simply connected open subsets U, V of X such that $X = U \cup V$ and $U \cap V$ is non-empty and simply connected, then X is simply connected.

Solution. Textbook, page 221. □

¶ **7.** Suppose that (X, a) and (Y, b) are path connected topological spaces and $f : (X, a) \to (Y, b)$ is a continuous mapping, and let $f_* : \pi_1(X; a) \to \pi_1(Y; b)$ be the induced homomorphism. Prove or give a counterexample.

(a) If f is injective, then f_* is injective.

(b) If f is surjective, then f_* is surjective.

(c) If f is bijective, then f_* is an isomorphism.

¶ **8.** Let S^n be the unit sphere in $\mathbf{R}^{n+1}$, and let $a = (0, 0, \cdots, 0, 1)$ be the north pole. For $n \geq 2$, prove the following.

(a) Any path $\alpha \in \prod(S^n; a, a)$ is homotopic relative to endpoints to a path β whose image does not fill the whole sphere, that is, such that $\beta([0, 1]) \not\subset S^n$.

(b) Any path $\alpha \in \prod(S^n; a, a)$ such that $\alpha([0, 1]) \not\subset S^n$ is homotopic relative to endpoints to the constant path c_a.

(c) S^n is simply connected.

Solution. (a) Any path in S^n is homotopic relative to endpoints to a polygonal path. If $n \geq 2$, the image of a polygonal path cannot be all of S^n.

(b) If p is any point in S^n, then $S^n \setminus \{p\}$ is homeomorphic to $\mathbf{R}^{n-1}$ (via stereographic projection for example), and therefore contractible.

(c) Any loop in S^n is homotopic to a loop in $S^n \setminus \{p\}$ relative to endpoints, and this is in turn homotopic to the constant loop.

□

¶ **9.** Prove that $X = \mathbf{R}^n \setminus \{0\}$ is simply connected if $n \geq 3$.

Solution. $\mathbf{R}^n \setminus \{0\}$ is homeomorphic to the product $S^{n-1} \times (0, \infty)$ via the mapping $x \mapsto \left(\dfrac{x}{|x|}, |x|\right)$.

□

¶ **10.** Let M be the space of obtained as the quotient space of $\mathbf{R}^2$ modulo the equivalence relation given by $(x, y) \sim (x', y')$ if and only if $x - x' = m$ is an integer and $y - (-1)^m y' = 0$. Prove that $\pi_1(M, a)$ is infinite cyclic.

Solution. The mapping $H : [0, 1] \times \mathbf{R}^2 \to \mathbf{R}^2$ given by $H(s, (x, y)) = (x, ty)$ is a deformation retraction of $\mathbf{R}^2$ onto the x-axis. It has the property that if $(x, y) \sim (x', y')$, then $H(x, y) \sim H(x', y')$. Therefore it induces a continuous mapping $H' : [0, 1] \times M \to M$. This H' is a deformation retraction of M onto the quotient space of the x-axis by the equivalence relation $x \sim x'$ if and only if $x' - x$ is an integer. This quotient is thus homeomorphic to a circle. Hence $\pi_1(M; a)$ is infinite cyclic. □

¶ **11.** Let $X = \mathbf{N} \cup \{\infty\}$, where $\mathbf{N}$ is the set of natural numbers, endowed with the topology in which $F \subset X$ is closed if and only if either $F = X$ or else F is a finite subset of $\mathbf{N}$. Prove that X is simply connected.

Solution. (a) Let m, n be any points in X. The mapping $\alpha : [0, 1] \to X$ given by $\alpha(0) = n$, $\alpha(1) = m$, and $\alpha(s) = \infty$ for $0 < s < 1$ is continuous. Indeed, if F is closed in X, then

$$\alpha^{-1}F = \begin{cases} \{0, 1\} & \text{if } n, m \in F, \\ \{0\} & \text{if } n \in F \text{ and } m \notin F, \\ \{1\} & \text{if } n \notin F \text{ and } m \in F, \\ \emptyset & \text{if } n \notin F \text{ and } m \notin F, \end{cases}$$

all of which are closed subsets of $[0, 1]$

(b) Let $\alpha, \beta : [0, 1] \to X$ be two loops based at ∞, that is, two paths such that $\alpha(0) = \beta(0) = \alpha(1) = \beta(1) = \infty$. Let $H : [0, 1] \times [0, 1] \to X$ be given by

$$H(s, t) = \begin{cases} \alpha(s), & \text{if } t = 0 \\ \beta(s), & \text{if } t = 1 \\ \infty, & \text{if } 0 < t < 1 \end{cases}$$

Then H is a homotopy from α to β relative to endpoints. Indeed, H has the correct boundary values, namely, $H(s, 0) = \alpha(s)$, $H(s, 1) = \beta(s)$ for all $0 \le s \le 1$, and $H(0, t) = H(1, t) = \infty$ for all $0 \le t \le 1$, and H is also continuous because if $F \subset X$ is closed and $F \ne X$, then $H^{-1}F = \alpha^{-1}F \cup \beta^{-1}F$ is also closed in $[0, 1] \times [0, 1]$.

□