### Homework 12. Solutions

1. Let $A \subset X$ be a retract, let $r : X \to A$ be the retraction mapping, and let $j : A \to X$ be the inclusion mapping. Prove that, for any $a$ in $A$, the homomorphism $r_* : \pi_1(X; a) \to \pi_1(A; a)$ is surjective, and the homomorphism $j_* : \pi_1(A; a) \to \pi_1(X; a)$ is injective.

**Solution.** The composition $r \circ j : A \to A$ is the identity mapping of $A$: $r \circ j = \text{id}_A$. Therefore, $(r \circ j)_* = r_* \circ j_*$ is the identity homomorphism of $\pi_1(A; a)$.

If $z \in \pi_1(A; a)$, then $j_*z \in \pi_1(X; a)$ and $r_*(j_*z) = z$, showing that $r_*$ is surjective.

If $z \in \pi_1(A; a)$ is such that $j_*z$ is the identity element of $\pi_1(X; a)$, then $r_*(j_*z) = z$ must be the identity element of $\pi_1(A; a)$, showing that $j_*$ is injective.

\[ \square \]

2. Prove that the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is not a retract of the plane $\mathbb{R}^2$.

**Solution.** By Problem 1, a retraction $r : \mathbb{R}^2 \to S^1$ induces a surjective homomorphism $r_* : \pi_1(\mathbb{R}^2; 1) \to \pi_1(S^1; 1)$. This is not possible because $\mathbb{R}^2$ is simply connected, hence $\pi_1(\mathbb{R}^2; 1)$ is the trivial group, while $\pi_1(S^1; 1)$ is infinite cyclic.

\[ \square \]

3. Prove that for a path connected space $X$ the following are equivalent:

(a) $X$ is simply connected;

(b) every continuous mapping $f : S^1 \to X$ is homotopic to a constant mapping.

(c) every continuous mapping $f : S^1 \to X$ admits a continuous extension to the closed disk $B^2$;

(d) any two paths $\alpha, \beta : [0, 1] \to X$ with the same initial and final points are homotopic relative to endpoints.

4. If $X$ is a topological space and $y$ is a path in $X$ from $\gamma(0) = a$ to $\gamma(1) = b$, then the mapping $\gamma_* : \pi_1(X; a) \to \pi_1(X; b)$ given by $\gamma_*[\alpha] = [\gamma^{-1} \alpha \gamma]$ is an isomorphism.

Prove that $\gamma_* = \gamma'_*$ for any paths $\gamma, \gamma' \in \Pi(X; a, b)$ if and only if $\pi_1(X, a)$ is commutative.

5. Let $(X, a)$ and $(Y, b)$ be pointed spaces. Prove that the fundamental group $\pi_1(X \times Y; (a, b))$ is isomorphic to the direct product $\pi_1(X; a) \times \pi_1(Y; b)$.

**Solution.** Textbook page 222.

6. Prove that if there are two simply connected open subsets $U, V$ of $X$ such that $X = U \cup V$ and $U \cap V$ is non-empty and simply connected, then $X$ is simply connected.

**Solution.** Textbook page 221.

7. Suppose that $(X, a)$ and $(Y, b)$ are path connected topological spaces and $f : (X, a) \to (Y, b)$ is a continuous mapping, and let $f_* : \pi_1(X; a) \to \pi_1(Y; b)$ be the induced homomorphism. Prove or give a counterexample.

(a) If $f$ is injective, then $f_*$ is injective.

(b) If $f$ is surjective, then $f_*$ is surjective.

(c) If $f$ is bijective, then $f_*$ is an isomorphism.

8. Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$, and let $a = (0, 0, \ldots, 0, 1)$ be the north pole. For $n \geq 2$, prove the following.

(a) Any path $\alpha \in \Pi(S^n; a, a)$ is homotopic relative to endpoints to a path $\beta$ whose image does not fill the whole sphere, that is, such that $\beta(\{0, 1\}) \not\subset S^n$.

(b) Any path $\alpha \in \Pi(S^n; a, a)$ such that $\alpha(\{0, 1\}) \subset S^n$ is homotopic relative to endpoints to the constant path $c_a$.

(c) $S^n$ is simply connected.

**Solution.** (a) Any path in $S^n$ is homotopic relative to endpoints to a polygonal path. If $n \geq 2$, the image of a polygonal path cannot be all of $S^n$.

(b) If $p$ is any point in $S^n$, then $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^{n-1}$ (via stereographic projection for example), and therefore contractible.

(c) Any loop in $S^n$ is homotopic to a loop in $S^n \setminus \{p\}$ relative to endpoints, and this is in turn homotopic to the constant loop.

\[ \square \]

9. Prove that $X = \mathbb{R}^n \setminus \{0\}$ is simply connected if $n \geq 3$.

**Solution.** $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to the product $S^{n-1} \times (0, \infty)$ via the mapping $x \mapsto \left(\frac{x}{|x|}, |x|\right)$.

\[ \square \]

10. Let $M$ be the space of obtained as the quotient space of $\mathbb{R}^2$ modulo the equivalence relation given by $(x, y) \sim (x', y')$ if and only if $x - x' = m$ is an integer and $y - (-1)^m y' = 0$. Prove that $\pi_1(M, a)$ is infinite cyclic.

**Solution.** The mapping $H : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $H(s, (x, y)) = (x, sy)$ is a deformation retraction of $\mathbb{R}^2$ onto the $x$-axis. It has the property that if $(x, y) \sim (x', y')$, then $H(x, y) \sim H(x', y')$. Therefore it induces a continuous mapping $H^* : [0, 1] \times M \to M$. This $H^*$ is a deformation retraction of $M$ onto the quotient space of the $x$-axis by the equivalence relation $x \sim x'$ if and only if $x' - x$ is an integer. This quotient is thus homeomorphic to a circle. Hence $\pi_1(M; a)$ is infinite cyclic.

\[ \square \]
11. Let \( X = \mathbb{N} \cup \{\infty\} \), where \( \mathbb{N} \) is the set of natural numbers, endowed with the topology in which \( F \subset X \) is closed if and only if either \( F = X \) or else \( F \) is a finite subset of \( \mathbb{N} \). Prove that \( X \) is simply connected.

Solution. (a) Let \( m, n \) be any points in \( X \). The mapping \( \alpha : [0, 1] \to X \) given by \( \alpha(0) = n \), \( \alpha(1) = m \), and \( \alpha(s) = \infty \) for \( 0 < s < 1 \) is continuous. Indeed, if \( F \) is closed in \( X \), then

\[
\alpha^{-1} F = \begin{cases} 
{0, 1} & \text{if } n, m \in F, \\
{0} & \text{if } n \in F \text{ and } m \notin F, \\
{1} & \text{if } n \notin F \text{ and } m \in F, \\
0 & \text{if } n \notin F \text{ and } m \notin F,
\end{cases}
\]

all of which are closed subsets of \([0, 1]\)

(b) Let \( \alpha, \beta : [0, 1] \to X \) be two loops based at \( \infty \), that is, two paths such that \( \alpha(0) = \beta(0) = \alpha(1) = \beta(1) = \infty \). Let \( H : [0, 1] \times [0, 1] \to X \) be given by

\[
H(s, t) = \begin{cases} 
\alpha(s), & \text{if } t = 0 \\
\beta(s), & \text{if } t = 1 \\
\infty, & \text{if } 0 < t < 1
\end{cases}
\]

Then \( H \) is a homotopy from \( \alpha \) to \( \beta \) relative to endpoints. Indeed, \( H \) has the correct boundary values, namely, \( H(s, 0) = \alpha(s), H(s, 1) = \beta(s) \) for all \( 0 \leq s \leq 1 \), and \( H(0, t) = H(1, t) = \infty \) for all \( 0 \leq t \leq 1 \), and \( H \) is also continuous because if \( F \subset X \) is closed and \( F \neq X \), then \( H^{-1} F = \alpha^{-1} F \cup \beta^{-1} F \) is also closed in \([0, 1] \times [0, 1]\). \( \square \)