Homework 11

¶ **1.** Prove the following:

(a) A space X is contractible if and only if X is homotopically equivalent to a singleton.

(b) Every contractible space is path-connected.

(c) Every retract in a contractible space is contractible. (Recall that $A \subset X$ is called a retract of X if there is a continuous map $r : X \to A$ such that $r(a) = a$ for every a in A.)

Solution. (a) According to the definition given in class, X is contractible if there is an x_0 in X such that the indetity mapping $\mathrm{id}_X : X \to X$ is homotopic to a constant mapping $c_{x_0} : X \to X$. This is another way of saying that X is homotopically equivalent to a singleton.

(b) By the above, if X is contractible, then there is a mapping $H : X \times [0,1] \to X$ such that $H(x,0) = x$ for all x in X and $H(x,1) = x_0$ for all x in X. Given x in X, the mapping $\alpha_x : [0,1] \to X$ given by $\alpha_x(t) = H(x,t)$, is a path from x to x_0. Therefore, given any x, y in X, $\alpha_y^{-1}\alpha_x$ is a path from x to y.

(c) Let $r : X \to A$ be the retraction mapping, let $a_0 = r(x_0)$, and let $K : A \times [0,1] \to A$ be the composition $r \circ H$. This mapping K is a homotopy of id_A to the constant mapping c_{a_0} because it is continuous and satisfies $K(a,0) = r(a) = a$ for all a in A and $K(a,1) = r(x_0) = a_0$ for all a in A. Thus A is contractible.

□

¶ **2.** (a) Prove that if $A \subset X$ is a deformation retract, then A and X are homotopically equivalent.

(b) Prove that any non-empty, compact, convex subset of $\mathbf{R}^m$ is a deformation retract of $\mathbf{R}^m$.

¶ **3.** A mapping $f : X \to Y$ is null-homotopic if it is homotopic to a constant mapping.
For a space X, let $CX = X \times [0,1]/X \times \{1\}$, the cone on X, obtained from the product $X \times [0,1]$ by identifying all points in the subset $X \times \{1\}$.

(a) Prove that a continuous mapping $f : X \to Y$ is null-homotopic if and only if f admits a continuous extension to CX.

(b) Prove that two null-homotopic mappings $f, g : X \to Y$ need not be homotopic to each other.

Solution. (a) Let $H : X \times [0,1] \to Y$ be a homotopy from f to a constant mapping c_{y_0}, for some $y_0 \in Y$. Then $H(x,0) = f(x)$ for all x and $H(x,1) = y_0$ for all x in X. Because H is constant on the subspace $X \times \{1\}$, it induces a continuous mapping $H' : CX \to Y$ which agrees with f on $X \times \{0\}$. Thus H' is an extension of f to CX.

(b) Let $Y = \{0,1\}$ be the two-point discrete space, let $X = \mathbf{R}$ and let $f, g : \mathbf{R} \to Y$ be $f(x) = 0$ for all x and $g(x) = 1$ for all x. The f and g are constant but are not null-homotopic because Y is not path connected.

□

¶ **4.** Prove or give counterexamples:

(a) If $f, g : X \to Y$ are homotopic and $A \subset X$, then the restrictions $f|A$ and $g|A$ are homotopic as maps $A \to Y$.

(b) If $f, g : X \to Y$ are homotopic and $B \subset Y$ is such that $f(X) \subset B$ and $g(X) \subset B$, then the f and g are homotopic as mappings from X into B.

(c) Two mappings f, g of a space X into a product space $Y = Y_1 \times \cdots \times Y_n$ are homotopic if and only if the compositions $\pi_k \circ f$ and $\pi_k \circ g$ are homotopic for all $k = 1, \cdots, n$.

(d) Two mappings f, g of a space X into a product space $Y = \prod_{\alpha \in A} Y_\alpha$ are homotopic if and only if the compositions $\pi_\alpha \circ f$ and $\pi_\alpha \circ g$ are homotopic for all $\alpha \in A$.

Solution. (a) True. If $H : f \simeq g$ is a homotopy and $j : A \to X$ is the inclusion and $i : [0,1] \to [0,1]$ is the identity mapping, then the composition $H \circ (j \times i) : f \circ j \simeq g \circ j$ is a homotopy of $f \circ j = f|A$ to $g \circ j = g|A$.

(b) False

(c) True. Compose with the projection mappings.

(d) True. Compose with the projection mappings.

□

¶ **5.** Let $f_1, g_1 : X \to Y$ and $f_2, g_2 : Y \to Z$. Suppose that there are homotopies $H_1 : f_1 \simeq g_1$ and $H_2 : f_2 \simeq g_2$, and let $H : X \times I \to Z$ be given by $H(x,t) = H_2(H_1(x,t),t)$. True or False: H is a homotopy from $f_2 \circ f_1$ to $g_2 \circ g_1$.

¶ **6.** Let (X, A) and (Y, B) be two pairs of spaces. A mapping $f : (X, A) \to (Y, B)$ is a continuous mapping $f : X \to Y$ such that $f(A) \subset B$.

Two mappings $f, g : (X, A) \to (Y, B)$ are homotopic relative to A if there is a continuous mapping $H : X \times [0, 1] \to Y$ such that

$$\begin{aligned} H(x, 0) &= f(x) \quad \text{for all } x \text{ in } X \\ H(x, 1) &= g(x) \quad \text{for all } x \text{ in } X \\ H(a, t) &= f(a) = g(a) \quad \text{for all } a \text{ in } A \text{ and all } t \text{ in } [0, 1] \end{aligned}$$

Prove that "being homotopic relative to A" is an equivalence relation on the set of all mappings $f : (X, A) \to (Y, B)$.

¶ 7. Let X be the subset of the plane $\mathbf{R}^2$ consisting of the line segment from $(0, 0)$ to $(1, 0)$ on the x-axis, the line segment from $(0, 0)$ to $(0, 1)$ on the y-axis, and the line segments $\{(1/n, y) \mid 0 \le y \le 1\}$. Let $A \subset X$ be the subspace $A = \{(0, 1)\}$.

(a) Prove that X is contractible.

(b) Prove that the identity $f = \mathrm{id}_X$ and the constant map $g = c_{(0,1)}$ are homotopic. (Here $c_{(0,1)}(x) = (0, 1)$ for all x.)

(iii) Prove that f and g are not homotopic relative to A.

Solution. Textbook Ch 3.3 Exercise 4, pages 121 and 221. □

¶ 8. Let D be an open subset of $\mathbf{R}^n$, let α be a path in D from x to y, and set $\delta = \inf\{|\alpha(s) - w| \mid w \in \partial D, 0 \le s \le 1\}$. Show that if β is any path in D from x to y and $|\alpha(s) - \beta(s)| \le \delta$ for all $0 \le s \le 1$, then α and β are homotopic relative endpoints.

Solution. Textbook Ch 3.2 Exercise 3, pages 118 and 221. □

¶ 9. A path α in $\mathbf{R}^m$ is a polygonal path if there is a partition $0 = s_0 < s_1 < \cdots < s_n = 1$ of the interval $[0, 1]$ such that $\alpha(s) = \dfrac{s_i - s}{s_i - s_{i-1}}\alpha(s_{i-1}) + \dfrac{s - s_{i-1}}{s_i - s_{i-1}}\alpha(s_i)$ on $[s_{i-1}, s_i]$, for $i = 1, \cdots, n$. Prove that any path in an open subset D of $\mathbf{R}^m$ is homotopic in D relative to endpoints to a polygonal path.

Solution. Textbook Ch 3.2 Exercises 4 and 5, pages 118 and 221. □

¶ 10. Let (X, d) be a compact metric space and let $a, b \in X$. Let $\prod(X; a, b)$ be the set of paths in X from a to b, endowed with the metric $D(\alpha, \beta) = \sup\{d(\alpha(s), \beta(s)) \mid 0 \le s \le 1\}$. Prove that two paths α, β in $\prod(X; a, b)$ are homotopic relative to endpoints if they are in the same path connected component of the metric space $\prod(X; a, b)$, that is, if there is a path in $\prod(X; a, b)$ from a to b.

Solution. Textbook Ch 3.2. Exercise 6, pages 118 and 221. □