¶ **1.** Prove that a Hausdorff space is locally compact if and only if every point has a compact neighborhood.

¶ **2.** (a) In a locally compact Hausdorff space, the intersection of an open set with a closed set is locally compact.

(b) A locally compact subset of a Hausdorff space is he intersection of a closed set and an open one.

(c) A dense subset of a compact Hausdorff space is locally compact if and only if it is open.

¶ **3.** (a) Prove that if X is locally compact and $f : X \to Y$ is continuous, open and onto, then Y is also locally compact.

(b) Prove that a non-empty product of of finitely many locally compact spaces is locally compact. (In general, a non-empty product is locally compact spaces if and only if all its factors are locally compact, and all but finitely many of them are compact.)

¶ **4.** Let X be Hausdorff and locally compact, and view X as a subspace of its one-point compactification X^*. Let $f : X \to \mathbf{R}$ be continuous. Prove that f admits a continuous extension to X^* (that is, there is $F : X^* \to \mathbf{R}$ continuous such that $F(x) = f(x)$ for all x in X) if and only if for each $\epsilon > 0$ there is a compact subset K_ϵ of X such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in X \setminus K_\epsilon$.

¶ **5.** A space X is regular if given any closed $F \subset X$ and any $x \in X \setminus F$, there are open disjoint subsets of X, one containing F and the other containing x. A space that is regular and T_1 is said to be a T_3 space (or to satisfy the T_3-axiom).
Prove that the following properties are equivalent for a topological space X.

(a) X is regular.

(b) If U is open in X and $x \in U$, there exists an open set V such that $x \in V^- \subset U$.

(c) each point has a neighborhood base consisting of closed sets.

¶ **6.** A space X is completely regular (or Tichonoff) if whenever F is a closed subset of X and $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f|F = 0$.
Sometimes it may be more convenient to use the following equivalent definition: given $x \in X$ and a neighborhood U of x, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus U) = 1$. A space that is completely regular and T_1 is said to be a $T_{3\frac{1}{2}}$-space (or to satisfy the $T_{3\frac{1}{2}}$-axiom).

(a) Prove that a subspace of a $T_{3\frac{1}{2}}$-space (Tichonoff space) is a $T_{3\frac{1}{2}}$-space (Tichonoff space).

(b) A nonempty product space is $T_{3\frac{1}{2}}$ (Tichonoff) if and only if each factor is $T_{3\frac{1}{2}}$ (Tichonoff).

¶ **7** (Moore plane)**.** Let M be the set of points $(x, y) \in \mathbf{R}^2$ with $y \geq 0$. Define a topology on M via neighborhoods as follows: A neighborhood of a point (x, y) in M with $y > 0$ is any subset of M that contains an open Euclidean disc centered at (x, y); a neighborhood of a point $(x, 0)$ is any subset of M that contains $(x, 0)$ and an open Euclidean disc contained in M that is tangent to $y = 0$ at $(x, 0)$.
Prove that M satisfies the $T_{3\frac{1}{2}}$-axion but it does not satisfy the T_4-axiom.

¶ **8.** Let X be a normal space. Let $A \subset X$ be a closed subset and $f : A \to \mathbf{R}$ continuous with $|f(x)| \leq c$ for each x in A. Prove that there is a continuous function $F : X \to \mathbf{R}$ such that

(a) $|F(x)| \leq c/3$ for all x in X.

(b) $|f(x) - F(x)| \leq 2c/3$ for all x in A.

Hint. Apply Uryshon's Lemma to the sets $f^{-1}[-c, -c/3]$ and $f^{-1}[c/3, c]$. Such function F is called a $1/3$-approximate extension of f.

¶ **9** (Tietze's Extension Theorem)**.** Let X be a normal space. Let A be a close subset of X and let $f : A \to [-1, 1]$ be a continuous function. Then there is $F : X \to [-1, 1]$ continuous such that $F(a) = f(a)$ for all a in A (F is called a continuous extension of f).
To prove this, let F_1 a $1/3$-approximate extension of f (as defined in Problem 8), and inductively let F_{n+1} be a $1/3$-approximate extension of $f - (F_1 + \cdots + F_n)$.

(a) Prove that $|f(x) - \sum_{i=1}^{n} F_i(x)| \leq (2/3)^n$ for all x in A.

(b) Prove that $|F_{n+1}(x)| \leq 2^n/3^{n+1}$ for all $x \in X$.

(c) Prove that the series $\sum_{n=1}^{\infty} F_n$ converges uniformly on X to a continuous function $F : X \to [-1, 1]$.

(d) Prove that F is a continuous extension of $f : A \to [-1, 1]$.

¶ **10.** Tietze's extension theorem was originally proved before Urysohn's lemma. Nowadays one proves Tietze's extension theorem as a corollary to Urysohn's lemma (as is done with Problem 8 and Problem 9 above. The question arises: can you prove Urysohn's Lemma as a corollary to Tietze's extension theorem?